Assignment 8

SLOB Mapped against the Module

To equip oneself with application-oriented knowledge of Assignment to facilitate management decisions for optimisation through resource allocation, managing competition, work scheduling and managing cost overrun, demand estimation, production and cost analysis etc.

Module Learning Objectives

After studying this module, the students will be able to:

- Recognise a problem of Assignment.
- State an Assignment problem in LP form.
- Solve Assignment problem using Hungarian Method
- Obtain solutions for the Special cases of Assignment like Maximisation Problem, Unbalanced Problem, Problem having Multiple Optimum Solution, Restricted Assignment Problem etc.
- Solve the problem related to Travelling Salesman using concepts of Assignment.

Assignment

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Assignment is a special case of Linear Programming where allocation of various resources (items) to various activities (receivers) is done on a one to one basis in such a way that the effectiveness is optimised. This can also be considered as a special case of Transportation in which the number of sources and destinations are equal. Also the capacity of each source as well as requirement of each destination is taken as 1. In fact Assignment is a degenerate problem of Transportation. Thus, an Assignment problem can be solved by both Simplex Method of solution of LPP and technique used for solution of Transportation problems. However both the methods are very cumbersome for solving an Assignment problem. These problems are solved by a simpler and more efficient method developed by Harold Kuhn in 1955 who named it as Hungarian Method to honour two Hungarian mathematicians D. Konig and J. Egevary whose earlier works was the basis of Kuhn`s work.

There are many situations where the assignment of people or machines etc. may be called for. Assignment of workers to machines, clerks to various check-out counters, salesmen to different sales areas, etc. are typical examples of these. The Assignment is a problem because people possess varying abilities for performing different jobs and therefore the costs of performing jobs by different people are different. Thus, in an assignment problem, the question is how the assignments should be made in order that the total cost involved is minimized.

Mathematical statement of the problem

Suppose there are `n` persons and `n` jobs and the assignment of the jobs to be done on one to one basis so that the total effectiveness (Total Cost or Time taken for completion of the jobs) is optimized (minimized). This problem can be stated in the form of (n × n) matrix, as shown below, for Cost or Effectiveness (C_{ij}), where C_{ij} is the Cost or Effectiveness associated with assigning ith person to the jth job.

$$
\begin{array}{ccc}
 & & \text{Jobs} \\
 & & \begin{array}{cccccccc} 1 & \quad 2 & \quad 3 & & & j & & n \end{array} \\
 & \begin{array}{c} 1 \\ 2 \\ 3 \\ \\ i \\ n \end{array} &
\left[\begin{array}{cccccccc}
C_{11} & C_{12} & C_{13} & \cdots & C_{ij} & \cdots & C_{1n} \\
C_{21} & C_{22} & C_{23} & \cdots & C_{2j} & \cdots & C_{2n} \\
C_{31} & C_{32} & C_{33} & \cdots & C_{3j} & \cdots & C_{3n} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
C_{11} & C_{12} & C_{13} & \cdots & C_{Cj} & \cdots & C_{in} \\
C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nj} & \cdots & C_{nn}
\end{array}\right] \\
\text{Persons} & &
\end{array}
$$

Mathematically and assignment problem can be stated as follows

Minimise the total cost

$$Z = \sum_{i=j}^{n} \sum_{j=1}^{n} C_{ij} x_{ij}$$

Subject to the constraints

(i) $\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2.........n$

which means each person should be assigned to one and only one job

(ii) $\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2.........n$

which means each job must be assigned to one and only one person.

(iii) x_{ij} = I, if i^{th} person is assgned to the j^{th} job

= 0, if i^{th} person is not assigned to the j^{th} job

The illustration given below is to facilitate understanding the method of formulating an Assignment problem as a Linear Programming problem.

Illustration 1

A computer centre has three expert programmers and needs to develop three application programs. The head of the certre estimates the time required (in minutes) by the respective experts to develop the application programs are as follows –

		Programmes		
		A	B	C
Programmers	1.	120	100	80
	2.	80	90	110
	3.	110	140	120

Formulate an LP Model.

Solution:

Key decision is to determine which Programmer should be assigned with which Program. Let us designate the assignment of the jth Programmer to the ith Program by the decision variable x_{ij} (where the subscripts i = A, B, C and j = 1, 2, 3)

Objective is to minimize the total time required i.e.. to

Minimize $Z = 120x_{11} + 100x_{12} + 80x_{13} + 80x_{21} + 90x_{22} + 110x_{23} + 110x_{31} + 140x_{32} + 120x_{33}$

Subject to the following constraints

1. Each Programmer must be assigned with development of one and only one Program. Thus,

   $x_{11} + x_{21} + x_{31} = 1$ for Program A, $x_{12} + x_{22} + x_{32} = 1$ for Program B, $x_{13} + x_{23} + x_{33} = 1$ for Program C

2. Each Program must be assigned to one and only one Programmer. Hence

   $x_{11} + x_{12} + x_{13} = 1$ for Programmer 1, $x_{21} + x_{22} + x_{23} = 1$ for Programmer 2, $x_{31} + x_{32} + x_{33} = 1$ for Programmer 3

3. $x_{ij} = 0$ or 1, for all i and j.

Hungarian Method

The following are the steps involved in the minimization of an assignment problem under this method:

Step 1 : Row Operation

Locate the smallest cost element in each row of the given cost table. Now subtract this smallest element from each element in that row. As a result, there shall be at least one zero in each row of this new table, called the reduced cost table.

Step 2 : Column Operation

In the reduced cost table obtained, consider each column and locate the smallest element in it. Subtract the smallest value from every entry in the column. As a consequence of this action, there would be at least one zero in each column of the reduced hole. Hence in each of the rows and columns of the second reduced cost table, there would be at least one zero.

Step 3 : Optimality

Draw the minimum no. of horizontal and vertical lines (not the diagonal ones) that are required to cover all the zero elements. If the no. of lines drawn is equal to the no. of rows/columns (i.e. order) of the given Cost Matrix, the solution is optimal and proceed to step 6. If the no. of lines drawn is less than the order of the given matrix then go to step 4.

Step 4 : Improved Matrix

Select the smallest uncovered (by the lines) cost element. Subtract this element from all uncovered elements including itself and add this element to each value located at the intersection of any two lines. The cost elements through which only one line passes remain unaltered.

Step 5 : Repeat step 3 and 4 until an optimal solution is obtained.

Step 6 : Given the optimal solution, make the job assignments as indicated by the 'zero' elements. This is done as follows:

(a) Locate a row which contains only one zero element. Assign the job corresponding to this element to its corresponding person. Cross out the zero's, if any, in the column corresponding to the element, which is indicative of the fact that the particular job and person are no more available.

(b) Repeat (a) for each of such rows which contain only one zero. Similarly, perform the same operation in respect of each column containing only one 'zero' element, crossing out the zero(s), if any, in the row in which the element lies.

(c) If there is no row or column with only a single 'zero' element left, then select a row/column arbitrarily and choose one of the jobs (or persons) and make the assignment. Thus, in such a case, alternative solutions exist

Illustration 2

An equipment under breakdown has five repair jobs to make it operative again. The Maintenance Manager of the organisation has assigned five mechanics of his department to do the jobs. The estimated time (hours) for each of the mechanics to carry out the jobs are given in the following table.

	Time required (Hours) to complete the Repair jobs				
Mechanic	A	B	C	D	E
I	7	5	9	8	11
II	9	12	7	11	10
III	8	5	4	6	9
IV	7	3	6	9	5
V	4	6	7	5	11

Assuming that each mechanic can be assigned to only one job, determine the minimum time assignment.

Solution:

Table showing supplied data

	Time required (Hours) to complete the Repair jobs				
Mechanic	A	B	C	D	E
I	7	5	9	8	11
II	9	12	7	11	10
III	8	5	4	6	9
IV	7	3	6	9	5
V	4	6	7	5	11

Minimum element of a row of the above table is subtracted from every element of that row and it is done for each row. The result is shown in the Table below.

Table – 1 showing reduced matrix after Row operation

	Time required (Hours) to complete the Repair jobs				
Mechanic	A	B	C	D	E
I	2	0	4	3	6
II	2	5	0	4	3
III	4	1	0	2	5
IV	4	0	3	6	2
V	0	2	3	1	7

Now the minimum element of a column of the above table is subtracted from every element of that column and it is done for each column. The result is shown in the Table below.

Table – 2 showing reduced matrix after Column operation

	Time required (Hours) to complete the repair jobs				
Mechanic	**A**	**B**	**C**	**D**	**E**
I	2	0	4	2	4
II	2	5	0	3	1
III	4	1	0	1	3
IV	4	0	3	5	0
V	0	2	3	0	5

Here we find that the minimum number of horizontal and vertical straight lines required to cover all the zero elements of the matrix = 4 ≠ Order (5) of the matrix. Hence the solution is non-optimal.

Thus, a new matrix table is formed as described in the following lines.

Minimum of all the elements which are not covered by the horizontal and vertical lines, drawn already, is found to be 1. This is subtracted from all the uncovered elements and added to the elements at the junction cells where a horizontal and a vertical line have intersected. Such cells are (V – B), (V – C) & (V – E). The result is shown in the next Table.

Table – 3 showing improved matrix (Optimal)

	Time required (Hours) to complete the repair jobs				
Mechanic	**A**	**B**	**C**	**D**	**E**
I	1	0	4	1	4
II	1	5	0	2	1
III	3	1	0	0	3
IV	3	0	3	4	0
V	0	3	4	0	6

Here we find that the minimum number of horizontal or vertical straight lines required to cover all the zero elements of the matrix = 5 = Order (5) of the matrix. Hence the solution is optimal

Now to make the assignments we start examining the rows one by one to see if there is any row with a single zero. Here the 1st row is having single zero at the cell (I – B). So we make an assignment here by putting a square boundary around the numerical figure zero at this cell. Correspondingly we check the column of this assigned cell to find if there is any other zero in it. We find a zero at the cell (IV – B) and we cross it out indicating no further assignment against B is possible. Similar activity is performed for the remaining rows, too and we get assignment at the cells (II – C), (III – D),(IV – E). and (V – A) The resultant matrix with assignments is shown in the Table below.

Table – 4 showing Optimal Assignments

	Time required (Hours) to complete the Repair jobs				
Mechanic	**A**	**B**	**C**	**D**	**E**
I	1	[0]	4	1	4
II	1	5	[0]	2	1
III	4	1	~~0~~	[0]	3
IV	4	~~0~~	3	4	[0]
V	[0]	3	4	~~0~~	6

Thus, the optimal solution is –

Repair job	Assigned to Mechanic	Time required (Hours)
A	V	4
B	I	5
C	II	7
D	III	6
E	IV	5
Total	-	27

So the minimum time required to complete all the Repair Jobs = 27 hours

Special cases in Assignment Problem

Some special cases of the Assignment Problem like, Maximisation Problem, Unbalanced Assignment Problem, Problems with Multiple Optimum Solution and Problems with Restriction are going to be discussed here.

1. Maximisation Problem

Though Hungarian Method is basically meant for solving Assignment Problems related to minimisation of Objective Function, but it can be effectively used for solving Assignment Problems related to maximisation also. Methodology of solving such a problem is as follows –

1. Convert the supplied Profit Matrix to a Relative Loss or Regret Matrix by subtracting the entry of each cell from the highest one among all the entries of the Matrix.
2. Now use Hungarian Method to minimise the Relative Loss.
3. Once assignments are available, find out maximum value of the total profit by considering the supplied values of Profit against the assigned cells.

Illustration 3

Five Salesmen are to be assigned to five Districts. Estimates of Sales Revenue (in `000 ₹) for each Salesman are given in the table below.

Districts	Salesman A	Salesman B	Salesman C	Salesman D	Salesman E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment pattern that maximises Revenue.

Solution:

This is a problem of Maximisation. So the given matrix has to be converted to a Relative Loss matrix by subtracting all the elements of it from the highest element 41. The resultant Matrix is shown below.

Table-1 showing Relative Loss matrix

Districts	Salesman A	Salesman B	Salesman C	Salesman D	Salesman E
1	9	3	1	13	1
2	1	17	13	20	5
3	0	14	8	11	4
4	19	3	0	5	5
5	12	8	1	6	2

Now Hungarian Method is applied and the results are shown in the following Tables so that Relative Loss is minimised.

Table-2 showing reduced matrix after Row operation

Districts	Salesman A	Salesman B	Salesman C	Salesman D	Salesman E
1	8	2	0	12	0
2	0	16	12	19	4
3	0	14	8	11	4
4	19	3	0	5	5
5	11	7	0	5	1

Table – 3 showing reduced matrix after Column operation

Districts	Salesman A	Salesman B	Salesman C	Salesman D	Salesman E
1	8	0	0	7	0
2	0	14	12	14	4
3	0	12	8	6	4
4	19	1	0	0	5
5	11	5	0	0	1

Here minimum number of horizontal and vertical straight lines to cover all the zeros = 4 ≠ Order (5) of the matrix. So the solution is non – optimal. An improvement over this one is done by subtracting the minimum of the uncovered elements i.e.. 1 from all the uncovered elements and adding that to the elements at the junction of lines

Table – 4 showing improved matrix (Non-optimal)

Districts	Salesman A	Salesman B	Salesman C	Salesman D	Salesman E
1	9	0	1	8	0
2	0	13	12	14	3
3	0	11	8	6	3
4	19	0	0	0	4
5	11	4	0	0	0

Here also minimum number of horizontal and vertical straight lines to cover all the zeros = 4 ≠ Order (5) of the matrix. So the solution is non – optimal. An improvement over this one is done by subtracting the minimum of the uncovered elements i.e.. 3 from all the uncovered elements and adding that to the elements at the junction of the horizontal and vertical lines. The resultant matrix is shown in Table- 5 below:

Table – 5 showing improved matrix (Optimal)

Districts	Salesman A	Salesman B	Salesman C	Salesman D	Salesman E
1	12	0	1	8	0
2	0	10	9	11	0
3	0	8	5	3	0
4	22	0	0	0	4
5	14	4	0	0	0

In this case we find that the minimum number of lines to cover the zeros = 5 = Order of the matrix. Hence the solution is optimal.

Neither any row nor any column of the matrix has single zero. Thus, we start assignment by arbitrarily choosing the first cell with zero in the 1st row i.e.. cell (1 – B) and other zero in the first row at cell (1 – E) is crossed out. Also zero appearing in the second column at the cell (4 – B) is crossed out.

Again 2nd Row is having two zeros. So arbitrarily first one of the two at cell (2 – A) is chosen and the other one at the cell (2 – E) is crossed out. Also zero appearing in first column at the cell (3 – A) is crossed out.

Now the 3rd Row is having single zero at the cell (3 – E). Here an assignment is done and correspondingly zero at the cell (5 – E) is crossed out.

Again 4th Row is having two zeros. So arbitrarily zero at the cell (4 – C) is chosen for assignment causing zeros at the cells (4 – D) and (5 – C) to cross out.

Now only one zero is left in the 5th Row and that is at the cell (5 – D) where the last assignment is done.

Table-6 showing Optimal Assignments

Districts	Salesman A	Salesman B	Salesman C	Salesman D	Salesman E
1	12	[0]	1	8	~~0~~
2	[0]	10	9	11	~~0~~
3	~~0~~	8	5	3	[0]
4	22	~~0~~	[0]	~~0~~	4
5	14	4	~~0~~	[0]	~~0~~

With the optimal assignments shown above, maximum total Sales Revenue is calculated using the supplied data against the assigned cells.

Calculation of Maximum Total Sales Revenue

Salesman	District assigned	Revenue (`000 ₹)
A	2	40
B	1	38
C	4	41
D	5	35
E	3	37
Total	-	191

So the maximum total Sales Revenue = ₹ 1,91,000

[Note – Presence of not a single row or column with single zero is indicative of Multiple Optimum Solution for the problem. It is discussed later on in this module.]

2. Unbalanced Assignment Problem

An Assignment Problem is said to be unbalanced if the number of its rows and columns are unequal. To solve such problems additional row or column (as the case may be) has to be introduced with all zero entries. In fact the introduction of additional row or column is needed to make the matrix square which is the prime requirement of an Assignment Problem. This newly introduced row or column is called Dummy Row or Dummy Column.

Methodology of the solution of such a problem is explained in the illustration below.

Illustration 4

A city corporation has decided to carry out road repairs on four main arteries of the city. The government has agreed to make a special grant of ₹ 50 lakhs towards the cost with a condition that the repairs must be done at the lowest cost and quickest time. If conditions warrant, then a supplementary token grant will also be considered favourably. The corporation has floated tender and five contractors participated in bidding. In order to expedite work, one road will be awarded to one contractor. The following matrix of Cost of Repairs is prepared by the corporation on the basis of the bids submitted by the participants.

	Cost of Repairs in ₹ Lakhs for			
Contractors	**Road 1**	**Road 2**	**Road 3**	**Road 4**
A	9	14	19	15
B	7	17	20	19
C	9	18	21	18
D	10	12	18	19
E	10	15	21	16

(i) Find the best way of assigning the repair work to the contractors and the total cost.

(ii) If it is necessary to seek supplementary grant then what should be the amount sought?

(iii) Which of the five contractors will be unsuccessful in his bid?

Solution:

(i) This is an unbalanced problem of Assignment where No. of rows (5) ≠ No. of columns (4) in the given cost matrix. To make the problem balanced we introduce a Dummy Column with all zero elements as shown below.

Table – 1 showing conversion of Unbalanced Cost Matrix to Balanced one

	Roads				
Contractors	**1**	**2**	**3**	**4**	**Dummy**
A	9	14	19	15	0
B	7	17	20	19	0
C	9	18	21	18	0
D	10	12	18	19	0
E	10	15	21	16	0

As each row of the above matrix contains a zero, it will remain unchanged after Row operation (1st step) of Hungarian Method. Thus, we straightway go for the next step of Column operation and arrive at the Table below.

Table – 2 showing reduced matrix after Column operation

	Roads				
Contractors	**1**	**2**	**3**	**4**	**Dummy**
A	2	2	1	0	0
B	0	5	2	4	0
C	2	6	3	3	0
D	3	0	0	4	0
E	3	3	3	1	0

So minimum no. of horizontal and vertical straight lines to cover all the zeros = 4 ≠ Order (5) of the matrix

Thus, the solution is non-optimal. An improvement of the solution is done by subtracting the minimum of the uncovered elements i.e.. 1 from all the uncovered elements and adding the same to the elements at the junction.

Table – 3 showing improved Matrix (Non-optimal)

	Roads				
Contractors	**1**	**2**	**3**	**4**	**Dummy**
A	**2**	**2**	**1**	**0**	**1**
B	**0**	**5**	**2**	**4**	**1**
C	**1**	**5**	**2**	**2**	**0**
D	**3**	**0**	**0**	**4**	**1**
E	**2**	**2**	**2**	**0**	**0**

Here also minimum no. of horizontal and vertical straight lines to cover all the zeros = 4 ≠ Order (5) of the matrix. Thus, the solution is non-optimal. An improvement of the solution is done by subtracting the minimum of the uncovered elements i.e.. 1 from all the uncovered elements and adding the same to the elements at the junction.

Table – 4 showing improved Matrix (Optimal)

	Roads				
Contractors	**1**	**2**	**3**	**4**	**Dummy**
A	**1**	**1**	**0**	**0**	**1**
B	**0**	**5**	**2**	**5**	**2**
C	**0**	**4**	**1**	**2**	**0**
D	**3**	**0**	**0**	**5**	**2**
E	**1**	**1**	**1**	**0**	**0**

Here minimum no. of horizontal and vertical straight lines to cover all the zeros = 5 = Order (5) of the matrix. So the solution is optimal.

Table – 5 showing Optimal Assignments

	Roads				
Contractors	**1**	**2**	**3**	**4**	**Dummy**
A	1	1	[0]	~~0~~	1
B	[0]	5	2	5	2
C	~~0~~	4	1	2	[0]
D	3	[0]	~~0~~	5	2
E	1	1	1	[0]	~~0~~

Row B is having only one zero at the cell (B -1). So it is assigned and zero in the same column at the cell (C -1) is crossed out. Now Row C is having one zero at the cell (C – Dummy). So it is assigned and zero in the same column in the cell (E – Dummy) is crossed out. Again Row E is having only one zero at the cell (E – 4). It is assigned and the zero in the cell (A - 4) of the same column is crossed out. Next we check columns to find single zero in column 2 at the cell (D – 2). It is assigned and zero at the cell (D – 3) of the same row is crossed out. Only the cell (A – 3) is left with a zero and it is assigned.

With the assignments shown above Minimum Total Cost of repairing the roads is calculated as follows -

Calculation of Minimum Total Cost

Road to be repaired	Assigned to Contractor	Cost of Repair (₹ Lakhs)
1	B	7
2	D	12
3	A	19
4	E	16
Total	-	54

So the best way of repairing the roads and associated cost is as shown in the above table

(ii) As the minimum total cost (₹ 54 Lakhs) exceeds sanctioned amount (₹ 50 Lakhs), a supplementary grant of ₹ 4 Lakhs to be sought.

(iii) From the Table 5 above we find that the Dummy Road is assigned to the Contractor C. So C will be unsuccessful in his bid because there is no existence of the Dummy road.

3. Assignment Problem with Multiple Optimum Solution or Alternative Solution

Such solutions exist if while assigning it is seen that neither any row nor any column is having single zero. In such case one has to firstly check the rows and then columns to find one with two zeros. One of these should be chosen arbitrarily and a square to be drawn around it to mark an assignment at that cell. The other zero in the same row or column should be crossed out. Alternatively the cell whose zero is bounded by drawing a square around it should be crossed out and the other one is assigned. This will lead to two Alternative Solutions. For situations with more than two zeros, the procedure is similar i.e.. one zero to be chosen arbitrarily and the others should be crossed out.

Illustration 3 above has multiple optimum solutions and the procedure to get those is shown below.

Illustration 5

Use the data of Illustration 3 above to find all the possible optimum solutions.

Solution:

To solve the sum we have to show all the steps done up to Table - 6 of Illustration 3. Thereafter the alternative solutions have to be shown in the Tables-7, 8 & 9 as follows.

Table-7 showing Alternative Optimal Assignments (2)

Districts	Salesman A	Salesman B	Salesman C	Salesman D	Salesman E
1	12	[0]	1	8	~~0~~
2	~~0~~	10	9	11	[0]
3	[0]	8	5	3	~~0~~
4	22	~~0~~	[0]	~~0~~	4
5	14	4	~~0~~	[0]	~~0~~

Table-8 showing Alternative Optimal Assignments (3)

Districts	Salesman A	Salesman B	Salesman C	Salesman D	Salesman E
1	12	[0]	1	8	~~0~~
2	~~0~~	10	9	11	[0]
3	[0]	8	5	3	~~0~~
4	22	~~0~~	~~0~~	[0]	4
5	14	4	[0]	~~0~~	~~0~~

Table-9 showing Alternative Optimal Assignments (4)

Districts	Salesman A	Salesman B	Salesman C	Salesman D	Salesman E
1	12	[0]	1	8	~~0~~
2	[0]	10	9	11	~~0~~
3	~~0~~	8	5	3	[0]
4	22	~~0~~	~~0~~	[0]	4
5	14	4	[0]	~~0~~	~~0~~

Calculation of maximum Total Sales Revenue (in `000 ₹) for all the Alternative Solutions are done as follows -

Calculation of Maximum Total Sales Revenue (`000 ₹) for various Alternatives

Alternative 1 (Table-6)*			Alternative 2 (Table – 7)			Alternative 3 (Table – 8)			Alternative 4 (Table – 9)		
Person	District	Sales	Person	District	Sales	Person	District	Sales	Person	District	Sales
A	2	40	A	3	41	A	3	41	A	2	40
B	1	38	B	1	38	B	1	38	B	1	38
C	4	41	C	4	41	C	5	40	C	5	40
D	5	35	D	5	35	D	4	36	D	4	36
E	3	37	E	2	36	E	2	36	E	3	37
Total	-	191	Total	-	191	Total	-	191	Total	-	191

[* For Table 6 refer to the solution of Illustration 3]

From above it is seen that the maximum total Sales Revenue in each case is ₹ 1,91,000. Thus, each solution gives same optimum result. In other words, the problem has multiple optimum solutions.

4. Problems with Restriction on Assignments or Prohibited Assignment

Sometimes in an Assignment Problem there may be a case when a particular resource (say, a person) cannot be assigned to a particular activity (say, a job). To handle such a problem a very high cost (or time which is to be minimised) is assigned to that cell of the matrix which is meant for this restricted or prohibited assignment. This automatically restricts any assignment at that cell. Very high cost is generally represented by ∞ or M.

Illustration 6

A company has taken on rent three floors (1st, 2nd and 3rd) of a multi storied building for their city office. It has been decided to locate Managers of Marketing, Purchase, HR, Finance and Company Secretary in the office. The management has earmarked in different floors five rooms having numbers 103, 201, 205, 302 and 304 for the above mentioned Managers. But no particular room has been allotted for any particular Manager and rather they have given option to indicate their preference of rooms so that decision can be taken by the management using some scientific method and subsequently arrangement of sitting of the subordinates of various Managers can be made. Managerial preferences are provided in the table below with 1st preference appearing in the top for each and every Manager.

Preference of Rooms of different Managers				
Marketing Manager	**Purchase Manager**	**HR Manager**	**Finance Manager**	**Company Secretary**
302	302	103	302	201
103	304	201	205	302
304	205	304	304	304
	201	205	103	
		302		

It is evident that most of the Managers have not given preference for all the available rooms because they feel that all the rooms do not have the facility they are looking for. Assuming that the preferences can be quantified by numbers, find out which manager should be assigned with which room to minimise the preferential measure.

Solution:

Let the preferential ranking of the rooms be quantified as 1, 2, 3, 4 and 5 as well as M for no preference. In fact the managers who have given no preference for certain rooms do not want those rooms to be allotted to them i.e.. assignment of certain Rooms to some Mangers are restricted. For this a very high value (M) is assigned to those specific cells of the matrix. Following table shows the numerical figures for various combinations of Manager and Room.

Table -1 showing Preferential Scores of different Rooms as awarded by various Managers

	Preferential scores of different Rooms by different Managers				
Managers	**Room -103**	**Room - 201**	**Room -205**	**Room - 302**	**Room -304**
Marketing Manager	2	M	M	1	3
Purchase Manager	M	4	3	1	2
HR Manager	1	2	4	5	3
Finance Manager	4	M	2	1	3
Company Secretary	M	1	M	2	3

Table -2 showing reduced matrix after Row subtraction operation

Managers	Room -103	Room - 201	Room -205	Room - 302	Room -304
Marketing Manager	1	M	M	0	2
Purchase Manager	M	3	2	0	1
HR Manager	0	1	3	4	2
Finance Manager	3	M	1	0	2
Company Secretary	M	0	M	1	2

Table – 3 showing reduced matrix after Column subtraction operation

Managers	Room-103	Room-201	Room-205	Room-302	Room-304
Marketing Manager	**1**	**M**	**M**	**0**	**1**
Purchase Manager	**M**	**3**	**1**	**0**	**0**
HR Manager	**0**	**1**	**2**	**4**	**1**
Finance Manager	**3**	**M**	**0**	**0**	**1**
Company Secretary	**M**	**0**	**M**	**1**	**1**

Here minimum number of horizontal and vertical straight lines to cover all the zeros = 5 = Order of the matrix. So the solution is optimal and assignments are made using standard procedure of Hungarian Method as below.

Table - 4 showing matrix with Optimal assignments

Managers	Room -103	Room - 201	Room -205	Room - 302	Room -304
Marketing Manager	1	M	M	[0]	1
Purchase Manager	M	3	1	~~0~~	[0]
HR Manager	[0]	1	2	4	1
Finance Manager	3	M	[0]	~~0~~	1
Company Secretary	M	[0]	M	1	1

Thus, the required assignment of Rooms should be –

Room 103 to HR Manager, Room 201 to Company Secretary, Room 205 to Finance Manager, Room 302 to Marketing Manager and Room 304 to Purchase Manager.

Problem of Travelling Salesman

This can be considered as a special case of Prohibited Assignment. Here a Salesman has to start his sales tour from a particular city of his territory and visit all the other cities within the territory in such a manner that he completes his tour at the same city from where he started. When the distance or time required to travel or cost of travel between the cities is known then the objective of the salesman is to schedule the tour in such a way that the total distance travelled or total time elapsed for the travel or total cost of travel is minimised. This type of problems can be solved by the algorithm used for Assignment.

Illustration 7

A travelling salesman has to visit five cities. He wishes to start from a particular city, visit each city once and then return to his starting point. The travelling cost (in `00 ₹) between any two cities is given in the table below

	To City				
From City	**A**	**B**	**C**	**D**	**E**
A	M	5	8	4	5
B	5	M	7	4	5
C	8	7	M	8	6
D	4	4	8	M	8
E	5	5	6	8	M

Find the cost minimising sequence of visit.

Solution:

Table – 1 showing reduced matrix after Row subtraction operation

	To City				
From City	**A**	**B**	**C**	**D**	**E**
A	M	1	4	0	1
B	1	M	3	0	1
C	2	1	M	2	0
D	0	0	4	M	4
E	0	0	1	3	M

Table – 2 showing reduced matrix after Column subtraction operation

	To City				
From City	**A**	**B**	**C**	**D**	**E**
A	**M**	**1**	**3**	**0**	**1**
B	**1**	**M**	**2**	**0**	**1**
C	**2**	**1**	**M**	**2**	**0**
D	**0**	**0**	**3**	**M**	**4**
E	**0**	**0**	**0**	**3**	**M**

Here minimum number of horizontal and vertical straight lines to cover all the zeros = 4 ≠ Order (5) of the matrix. So the solution is non–optimal. Improvement of the above matrix is done by subtracting the minimum value of the uncovered elements i.e. 1 from all the uncovered elements and adding the same to the elements at the junction of the horizontal and vertical lines. The resultant matrix is shown below.

Table – 3 showing improved matrix (Optimal)

	To City				
From City	**A**	**B**	**C**	**D**	**E**
A	**M**	**0**	**2**	**0**	**1**
B	**0**	**M**	**1**	**0**	**1**
C	**1**	**0**	**M**	**2**	**0**
D	**0**	**0**	**3**	**M**	**5**
E	**0**	**0**	**0**	**4**	**M**

Here the minimum number of straight lines required to cover all the zeros = 5 = Order of the matrix. So the solution is optimal. Now assignments are done by following the standard rules of Hungarian Method as below.

Table – 4 showing matrix with Optimal Assignments (Alternative – 1)

	To City				
From City	**A**	**B**	**C**	**D**	**E**
A	M	[0]	2	~~0~~	1
B	~~0~~	M	1	[0]	1
C	1	~~0~~	M	2	[0]
D	[0]	~~0~~	3	M	5
E	~~0~~	~~0~~	[0]	4	M

As per the solution above, the Salesman will travel from A to B, then B to D, then D to A. But this is not meeting the requirement of travelling through all the cities and finally returning to the starting point i.e.. A. Hence the solution is unacceptable.

Table – 5 showing matrix with Optimal Assignments (Alternative – 2)

	To City				
From City	**A**	**B**	**C**	**D**	**E**
A	M	~~0~~	2	[0]	1
B	[0]	M	1	~~0~~	1
C	1	~~0~~	M	2	[0]
D	~~0~~	[0]	3	M	5
E	~~0~~	~~0~~	[0]	4	M

Again the solution above shows the travelling route as A to D, then D to B, then B to A. This is also not acceptable because of violation of the basic requirement of the problem.

Under the circumstances it is decided to try for the assignment at the cells which are having next highest entry after zero. It can be mentioned that as far as practicable Assignments should be done at the cells having 0 entry. Here next highest entry in the table after 0 is 1 and it appears at the four cells – (A, E), (B, C), (B, E) and (C, A). By arbitrarily choosing any one of these, assignment is done afresh. Let the cell (A, E) be chosen for the purpose

Table – 5 showing matrix with the required solution

	To City				
From City	**A**	**B**	**C**	**D**	**E**
A	M	~~0~~	2	~~0~~	[1]
B	~~0~~	M	1	[0]	1
C	1	[0]	M	2	~~0~~
D	[0]	~~0~~	3	M	5
E	~~0~~	~~0~~	[0]	4	M

Using the standard procedure of Hungarian Method assignments are made starting from 1st Row cell (A, E) and finally required solution is reached which shows the travel route of the salesman as – A to E, E to C, C to B, B to D and D to A. Minimum Cost of travel is 5 + 4 + 7 + 4 + 6 = 26 ('00 ₹)

Illustration 8 (Case Study involving assignment of three items)

Sugarcane is a globally important commercial crop that can be used to produce both direct and indirect products such as Sugar, Ethanol, Fodder etc. Sugarcane is a tall perennial grass reaching 3 to 4 metres in height. Its cultivation is not easy and it needs special tools to be harvested effectively.

To harvest sugarcane, cutting at the ground level is needed because the sweetness level is higher in this part of the cane. The tips of the stem and the leaves are immediately removed. Traditionally hand harvesting is effective for this. Currently mechanical harvesters play an important role in sugarcane harvesting due to shortage of labour and increased labour costs. A harvester is very powerful since it can replace 200 labourers. In the first era of using the harvester instead of human labour, the price of the harvester is still high. The number of harvesters in use is limited but there are many farmers who want to use it. Therefore the scheduling of harvesters is needed to achieve the aim of the farmers or the owner of the harvester.

Currently many brands, types and sizes of harvesters are available in the market. There is limitation for using a particular harvester such as some fields cannot be served by some harvester types due to the steepness of the field or insufficient roads to the field. This makes it more difficult to assign the field to the suitable harvester. Moreover, the cost control is also of interest to the owner of the harvester. The level of effectiveness of the machine has been considered when assigning the harvester to the field. It depends on a few factors, such as the model and number of operating years as well as experience of the user.

One harvester can be assigned to more than one field if it has enough working time available even after serving one field. Different harvesters have to work for different hours in a field because of the variation in their harvesting capacities. Each harvester has limitation on it`s available working time. Therefore assigning harvesters to different fields falls under the category of generalized assignment problem or GAP.

As mentioned above, besides assigning harvesters to fields there are various other factors to be considered, too. So the problem here cannot be taken as a simple GAP. One of the important factors is assignment of drivers to harvesters. The skill level of the driver will affect the effectiveness of the harvester, which also has a different capacity. The capacity of the harvester in this case, refers to the driving and harvest speeds which have great influence on the resource usage. Then the fields will be assigned to the harvesters which will affect the profit because each field has different sweetness and density levels. Actually this determines the amount of sugarcane that can be harvested per day. Also it has been mentioned before that one harvester can be assigned to more than one field depending on it`s working time availability. Thus, instead of considering it as a GAP, the problem should be taken as a specialised version of GAP.

Thus, the problem is composed of three actors – the driver, the harvester and the sugarcane field. Here a driver is to be assigned to a harvester set which is to be paired to a suitable field for harvesting so that the daily profit is maximized. The profit depends on the income and cost generated in the system, while the income is generated from the amount of harvested sugarcane sold multiplied by the sweetness level of the harvested sugarcane (the price of sugarcane sold is based on per ton unit). The profit is the result of total income subtracted by the cost of the harvester`s fuel consumption and the corresponding driver`s daily wages. There is a pool of drivers with various levels of experience and wages. The harvester sets have a range of machine types and operating years, which determines the fuel consumption rate. The efficiency of harvesting is also affected by the driver of the particular harvester set. After the harvester set and it`s driver have been assigned, they will start to harvest the assigned sugarcane field. Each sugarcane field has a different size, sweetness level and travelling distance. All these activities generate a special case of GAP. So the solution of the problem will not be an exact one and rather it will be a heuristic one.

A differential evolution (DE) algorithm is proposed to be used for arriving at a solution to the problem. The steps involved in the proposed DE are – (1) Generate initial solution, (2) Implement the mutation process, (3) Execute the recombination process & (4) Complete the selection process. Generally DE has three types of vectors, namely Target Vectors, Mutant Vectors and Trial Vectors. Also the DE Algorithm works on the real numbers generated randomly by the DE operators.

Assuming that there are 6 Fields of sugarcane, 4 Harvesters and 5 Drivers as well as 5 sets of target vectors, the initial set of randomly generated numbers for finding an Initial Solution are given as follows –

Target Vector	Sugarcane Field						Harvester				Driver of the Harvester				
	1	2	3	4	5	6	1	2	3	4	1	2	3	4	5
1	0.08	0.61	0.24	0.22	0.21	0.45	0.72	0.09	0.43	0.47	0.65	0.36	0.63	0.00	0.32
2	0.71	0.45	0.40	0.63	0.78	0.07	0.61	0.16	0.54	0.12	0.56	0.70	0.71	0.70	0.38
3	0.55	0.20	0.63	0.34	0.39	0.14	0.10	0.93	0.61	0.09	0.06	0.35	0.88	0.92	0.06
4	0.80	0.75	0.49	0.76	0.74	0.55	0.18	0.82	0.90	0.57	0.75	0.44	0.86	0.19	0.94
5	0.22	0.20	0.12	0.42	0.74	0.90	0.72	0.09	0.43	0.47	0.65	0.36	0.63	0.00	0.32
A_K	80	50	40	70	32	41									
B_K	1.2	1.5	0.9	0.8	1.0	0.8									
E_J							120	180	110	130					
U_J							1.1	1.3	1.0	1.2					
T_J							9	10	6	8					
R_i											1.2	0.7	0.8	1.1	1.2
P_i											0.9	1.2	1.1	0.9	0.8

The symbols used in the above table have the following meaning –

A_K = Area of the sugarcane field K in Acres & B_K = Sweetness factor of the crop of field K

E_J = Fuel consumption rate of harvester J in ₹/hour, U_J = Age factor of harvester J & T_j = Harvesting rate of harvester in Acres /hour

R_i = Experience factor of driver i (which affects fuel consumption rate) & P_i = Experience level of the driver i (which affects harvesting rate)

Time required in hour to drive the harvester J from the Parking Area to the sugarcane field K are given in the table below –

Harvester	Sugarcane Field					
	1	2	3	4	5	6
1	0.80	0.08	0.99	0.35	0.89	0.15
2	0.32	0.79	0.11	0.04	0.18	0.55
3	0.84	0.65	0.66	0.70	0.08	0.05
4	0.25	0.95	0.08	0.63	0.89	0.21

Additional information is as follows –

1. Fuel cost is ₹115 per hour,
2. Selling price of the sugarcane is ₹600 per ton,
3. 12 tons of standard sugarcane is obtained from 1 Acre of land,
4. All drivers are allowed to work for 9 hours a day,

Use the following methodology to find the Initial solution of the problem

1. Sort all the subsets of the vectors (i.e. Field, Harvester & Driver) in ascending order.
2. Assign a driver to a harvester and subsequently assign the harvester to the sugarcane field/s using the order obtained in step (1) above.

While doing this, the following rule should be used –

(a) Assign the entity (Field, Harvester, Driver) that is in position at the front of the order first

(b) If the current position violates the capacity constraint of the harvester (Maximum 9 hours of usage per day) the entity that is in the next position is allowed to be replaced.

Do you think there is sufficient resource to harvest all the six Sugarcane Fields?

Solution:

As per the given instructions of the problem, we have to first of all sort all the subsets of the vectors in ascending order. The results after sorting the data corresponding to the Target Vector 1 is given in the table below.

Target Vector	Sugarcane Field						Harvester				Driver of the Harvester				
	1	5	4	3	6	2	2	3	4	1	4	5	2	3	1
1	0.08	0.21	0.22	0.24	0.45	0.61	0.09	0.43	0.47	0.75	0.00	0.32	0.36	0.63	0.65

The sequence of Assignment in this case is – (1) Pairing of Driver & Harvester and (2) Pairing of Field with the selected set of Driver & Harvester

As per the table above the first pair of Driver & Harvester should be 4 (1st one of the Driver`s order) and 2 (1st one of the Harvester`s order)

Real harvesting rate of Harvester 2 = Harvesting rate of Harvester 2 × Experience level of Driver 4 = $T_2 \times P_4$

= 10 × 0.9 = 9 Acres / Hour

Now this pair of Driver (4) & Harvester (2) should be assigned to Sugarcane field 1 (1st one of the Field`s order in the above table)

From the given data in the question we have,

Time required to drive Harvester 2 from the parking area to the Sugarcane Field 1 = 0.32 hour

As each Harvester can work at the most 9 hours a day, the effective working hours for Harvester 2

= 9 – (0.32 × 2) = 8.36

So the combination of Driver 4 & Harvester 2 is capable of harvesting at the most (8.36 × 9) = 75.24 Acres / day

But the Area of Sugarcane Field 1 = 80 Acres which is more than 75.24 Acres

Thus, the combination of Driver 4 & Harvester 2 cannot be assigned to Field 1.

So the combination should be tried for the next Field in the order – Field 5 which has an Area of 32 Acres

Also Harvester 2 requires 0.18 hour to be driven from parking area to Field 5. So effective working hours for this Harvester = 9 – (0.18 × 2) = 8.64

Thus, the combination of Driver 1 & Harvester 2 can harvest at the most (8.64 × 9) = 77.76 Acres which is more than 32 Acres by 77.76 – 32 = 45.76 Acres

So Driver 4 & Harvester 2 combination can be utilised to harvest another field.

Next field in the order is Field 4 with an Area of 70 Acres which is much more than the balance capacity of 45.76 Acres of the particular Driver – Harvester combination. But the next field in the order i.e. Field 3 has an Area of 40 Acres. Hence it can be assigned to the Driver – Harvester combination of 4 – 1.

Thus, the combination of Driver 4 & Harvester 2 should be assigned to the Fields 5 and 3.

Following same technique other combinations of Driver and Harvester are assigned to the remaining Fields. Results of the assignment are shown below.

Driver	Experience level of Driver	Harvester	Harvesting Capacity (Acres / hour)	Actual Harvesting Capacity (Acres/ hour)	Assigned Sugarcane Field	Area of the field (Acres)	Time required to harvest & movement (hours)	Total time during which Harvester is busy
(1)	(2)	(3)	(4)	(5) = (2) × (4)	(6)	(7)	(8) = [(7) ÷ (5)] + Movement time	(9)
4	0.9	2	10	9	5	32	3.55 + 0.36 = 3.91	3.91 + 4.66 = 8.57 Hrs.
					3	40	4.44 + 0.22 = 4.66	
5	0.8	3	6	4.8	6	41	8.54 + 0.10 = 8.64	8.64 Hrs.
2	1.2	4	8	9.6	1	80	8.33 + 0.50 = 8.83	8.83 Hrs.
3	1.1	1	9	9.9	4	70	7.07 + 0.70 = 7.77	7.77 Hrs.

[**Note** – Movement time refers to the to and fro time required to drive a Harvester from parking area to the Field]

From the column no. (9) of the above table it is seen that all the assigned Harvesters are remaining busy for a period less than 9 hours. Hence the time constraint is not violated.

But Field No. 2 could not be assigned due to non-availability of Harvester. Hence there is insufficient resource to harvest all the six fields as far as the initial solution is concerned.

EXERCISE

A. Theoritical Questions

Multiple Choice Questions

1. Which of the following methods is used to solve the Assignment problems?

   (a) Stepping Stone Method

   (b) Hungarian Method

   (c) North West Corner Method

   (d) Vogel's Approximation Method

2. Assignment of work to men and machines is known as

   (a) Scheduling

   (b) Loading

   (c) Balancing of Line

   (d) None of these

3. In an Assignment matrix of size (5 × 5), the total number of decision variables in the objective function is –

   (a) 10

   (b) 5

   (c) 25

   (d) 15

4. An Assignment problem is solved to minimise the total time required to complete three jobs on three different machines such that each job is processed by exactly one machine and each machine processes exactly one job. The minimum total processing time is found to be 480 minutes. After a few days of operation, there has been a change in the design of the second job. Due to this, the processing time of the second job is increased by 15 minutes in either of the machines. The revised minimum total processing time will be –

   (a) 495 minutes

   (b) 465 minutes

   (c) 480 minutes

   (d) None of these

5. Assignment problem can be considered as a particular case of -

   (a) Transportation problem

   (b) Sequencing problem

   (c) Queuing problem

   (d) All of these

6. Dummy row or column is added in an assignment problem –

   (a) To prevent a solution to become degenerate.

   (b) To reduce the total cost of assignment.

   (c) To increase the profit function.

   (d) To balance total activities and total resources

7. While solving an assignment problem, an activity is assigned to a resource with zero opportunity cost because objective is to –

   (a) Reduce total cost of assignment to zero.

   (b) Reduce cost of that assignment to zero.

   (c) Minimise total cost of assignment.

   (d) Maximise total cost of assignment.

8. In an assignment problem –

   (a) First activity is assigned to first resource

   (b) Any number of activities can be assigned to each resource.

   (c) It depends on how many resources are available.

   (d) Only one activity be assigned to each resource.

9. An assignment problem can be viewed as a special case of transportation problem in which the capacity from each source is ____ and the demand at each destination is ____.

   (a) Unlimited, unlimited

   (b) One, unlimited

   (c) One, one

   (d) Unlimited, one

10. In marking assignments which of the following should be preferred?

   (a) Only row having single zero

   (b) Only column having single zero

   (c) Column having more than one zero

   (d) Only row / column having single zero.

11. The assignment matrix is always a ____

   (a) Rectangular matrix

   (b) Identity matrix

   (c) Square matrix

   (d) None of these

12. Maximisation assignment problem is transformed into a minimisation problem by _____

   (a) Adding each entry of a column to the maximum value of that column

   (b) Subtracting each entry in a column from maximum value in that column.

   (c) Subtracting each entry of the table from the maximum value of the table.

   (d) Adding each entry of the table to the maximum value in the table.

13. The assignment problem will have alternative solutions when it has ___

   (a) At least one zero in any row or column

   (b) All rows have two zeros.

(c) Two diagonal elements are zeros

(d) None of the above.

14. In the Hungarian Method of solving Assignment problem, the row reduction is obtained by

(a) Dividing each row by the elements of the row above it.

(b) Subtracting the elements of the row from the elements of the row above it.

(c) Subtracting the smallest element from all other elements of the row.

(d) Subtracting all the elements of the row from the highest element in the matrix.

15. The horizontal and vertical lines drawn to cover all zeros of the total opportunity matrix for an optimal solution must be –

(a) Equal to m x n, where m = No. of rows & n = No. of columns.

(b) Equal to each other.

(c) Equal to m + n, where m = No. of rows & n = No. of columns

(d) Equal to the Order of the matrix.

16. In a problem of Travelling Salesman, the diagonal elements of the matrix from top left corner are all –

(a) Zeros

(b) Negative

(c) Ones

(d) Infinitely large

17. The similarity between Assignment Problem and Transportation Problem is –

(a) Both are rectangular matrices

(b) Both are square matrices

(c) Both can be solved by graphical method

(d) Both have objective function and non-negativity constraints.

18. When we try to solve the Assignment problem by Transportation algorithm the following difficulty arises.

(a) There will be a tie while making allocations.

(b) The problem will get alternate solution.

(c) The problem degenerates and we have to use epsilon to solve degeneracy.

(d) The Assignment problem cannot be solved by Transportation algorithm.

19. The following character dictates that the Assignment matrix is a square one.

(a) The allocations in Assignment problem are one to one.

(b) Because we find row opportunity cost matrix.

(c) Because we find column opportunity cost matrix.

(d) Because after making allocations, horizontal and vertical lines are to be drawn.

20. An Assignment problem is considered as a special case of Transportation problem because –

(a) The number of rows is equal to the number of columns

(b) All x_{ij} = 0 or 1

(c) All rim conditions are equal to 1

(d) All of these

21. An Assignment problem can be solved by –

(a) Simplex method

(b) Transportation method

(c) Both (a) and (b)

(d) Only (b) but not (a)

22. The Hungarian Method for solving an Assignment problem can also be used to solve –

(a) Transportation problem

(b) Travelling Salesman problem

(c) Both (a) and (b)

(d) Not (a) but (b)

23. A firm is required to procure three items I, II & III from three vendors V_1, V_2 & V_3 respectively. The quoted prices in Rupees are given in the table below. The management policy clearly states that each item should be procured from only one vendor and each vendor should supply only one item. The minimum total cost of procurement is –

	VENDORS		
ITEMS	V_1	V_2	V_3
I	110	120	130
II	115	140	140
III	125	145	165

(a) ₹ 375

(b) ₹ 385

(c) ₹ 390

(d) None of the above

24. In a machine shop four jobs need to be assigned to four machines. Each of the jobs is to be assigned to one machine only at a time. The time in minutes required to complete different jobs in different machines is given in the table below.

	MACHINES			
JOBS	I	II	III	IV
A	15	13	14	17
B	11	12	15	13
C	13	12	10	11
D	15	17	14	16

In order to ensure that the total time to complete the jobs is minimum, the optimal assignment of the jobs is –

(a) A to IV, B to II, C to III and D to I

(b) A to II, B to I, C to IV and D to III

(c) A to II, B to I, C to III and D to IV

(d) A to IV, B to II, C to I and D to III

25. If there are n jobs and n workers, there would be –

(a) n! solutions

(b) (n – 1)! solutions

(c) (n!).n solutions

(d) n solutions

26. The Assignment problem

(a) Requires that only one activity be assigned to each resource

(b) Is a special case of Transportation problem

(c) Can be used to maximise the resources

(d) All of the above

27. To proceed with the MODI algorithm for solving an assignment problem, the number of dummy allocations need to be added are –

(a) n

(b) n -1

(c) 2n

(d) 2n – 1

28. An optimal solution of an assignment problem can be obtained only if –

(a) Each row and column has only one zero element

(b) Each row and column has at least one zero element

(c) Both the diagonals of the matrix have zero element

(d) None of the above

29. The procedure used to solve Assignment problems wherein one reduces the original assignment costs to a table of opportunity costs is called _____

(a) Stepping Stone Method

(b) Matrix Reduction

(c) MODI Method

(d) Northwest Reduction

30. When a maximisation assignment problem is converted to minimisation problem, the resultant matrix is called

(a) Cost matrix

(b) Profit matrix

(c) Regret matrix

(d) Dummy matrix

Answers

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
b	b	c	a	a	d	c	d	c	d	c	c	d	c	d
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
d	d	c	a	d	c	d	b	b	a	d	b	d	b	c

State True or False

1. In the optimal solution of Travelling Salesman problem all the assignments might not occur at the cells having zero entries.
2. As Assignment is a Linear Programming Problem, it can be solved using Graphical Method.
3. Square type cost matrix is the prime requirement of an Assignment Problem. Thus, an Assignment Problem with unequal number of rows and columns is not possible.
4. It is true that an Assignment problem can be solved using Transportation algorithm, but the reverse is not true.
5. To test optimality of an Assignment solution we have to cover all the zero entries of the matrix by only horizontal or only vertical straight lines.
6. While formulating an Assignment problem, described in a matrix of order 3, as a Linear Programming problem the total number of decision variables should be nine.
7. Hungarian Method is applicable for solving only the minimisation problems.
8. For problems with preferred assignment, we start with zero entries at the cells having preference.
9. If a dummy column is introduced to make an assignment problem balanced, then after row subtraction the matrix remains same as before.
10. Assignment problems are more frequently solved by Simplex Method.
11. An improvement of a non-optimal assignment solution is done by subtracting all the uncovered elements from the highest uncovered element and adding the same at the junction elements.
12. After getting an optimal solution, the first step to start assignment is to find out a row with a single zero entry.
13. While making an assignment at a cell with zero entry, the other cells with zero entry in the corresponding row or column should be crossed out because that is what we are supposed to do.
14. Assignment matrix of order 'n' yields only 'n' allocations where as a non-degenerate solution of Transportation demands (2n – 1) cell allocations. Thus, solution of Assignment problem by Transportation method is not possible.

15. In an optimal solution, while assigning cells arbitrary choice of zero entry cell is necessary when not a single row or column is available with only one zero.

Answers:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
T	F	F	T	F	T	T	F	T	F	F	T	F	F	T

- **Fill in the blanks**

1. When a particular assignment in the given problem is not possible or restricted as a condition, it is called a _______ problem.
2. Assignment problem is a special case of Linear Programming and can be solved by _______ method.
3. The method of solving Assignment problem is called Hungarian to honour two Hungarian mathematicians based on whose earlier work _______ developed the same in 1955.
4. If the number of rows and columns are not equal in an Assignment matrix then it is called ______ problem.
5. The first step of solution of Assignment problem by Hungarian method is ______ subtraction operation.
6. The _______ test of Hungarian Method refers to checking whether the minimum number of horizontal and vertical straight lines to cover all the zero entries of the matrix is equal to the order of the matrix or not.
7. For a non-optimal solution of Assignment problem, the improvement of solution matrix is done by subtracting the _______ of the uncovered elements from all the uncovered elements and adding the same to the elements at the junctions of the horizontal and the vertical lines.
8. Assignment problem is a degenerate type _____ problem.
9. For an unbalanced Assignment problem, introduction of a ____ row or column is necessary.
10. In case of Assignment problem with multiple optimum solutions, the minimum value for all the solutions is ________ .
11. All the entries in a dummy row or column are considered to be ________.
12. The table shows ________ solution of an Assignment problem.

	A	B	C
1.	40	10	0
2.	0	0	30
3.	0	20	10

13. For the given Assignment Matrix, the number of zeros after row subtraction operation will be ____

	A	B	C
1.	120	100	80
2.	80	90	110
3.	110	140	120

14. In the problem of Travelling Salesman, the basic requirement is a ____ trip of the person.
15. If the matrix shown in Q. No. (13) above is for a problem of maximisation then all the elements of it should be subtracted from the element ___ to get a regret matrix.

Answers:

1.	**Prohibited**	**2.**	**Simplex**
3.	**Kuhn**	**4.**	**Unbalanced**
5.	**Row**	**6.**	**Optimality**
7.	**Minimum**	**8.**	**Transportation**
9.	**Dummy**	**10.**	**Same**
11.	**Zero**	**12.**	**Optimal**
13.	**3**	**14.**	**Round**
15.	**140**		

Short essay type questions

1. Why is it necessary to cross out the zeros of the corresponding rows and columns of a solution matrix once an assignment is made to a particular cell?
2. Where from the name Hungarian is derived for the method of solution of Assignment problems?
3. Describe the beginning step of solution of an unbalanced problem of Assignment.
4. Can an Assignment problem ever be a non-degenerate transportation problem?
5. Give examples of two areas of application of Assignment problem.

Essay type questions

1. Give an example to show that an Assignment problem can be formulated as a Linear Programming Problem.
2. How will you handle the following situations in an Assignment problem? Write in brief (i) Maximisation of the Objective Function, (ii) Impossible Assignment
3. What is Assignment problem? Discuss its method of solution.

B. Numerical Questions

Comprehensive Numerical Problems

1. A Methods Engineer wants to assign four new methods to three work centres. The assignment of the new methods will increase production and they are given in the matrix below. If only one method can be assigned to a work centre, determine the optimum assignment.

	Increase in production (Units) at the Work Centres		
Methods	A	B	C
1	10	7	8
2	8	9	7
3	7	12	6
4	10	10	8

Is this a problem of multiple optimum solution? If yes, then find the alternative solution/s also.

2. Consider a problem of assigning four junior assistants to four tasks. The time (hours) required to complete the tasks are given in the table below.

	TASKS			
CLERKS	A	B	C	D
1	4	7	5	6
2	-	8	7	4
3	3	-	5	3
4	6	6	4	2

Assistant 2 cannot be assigned to task A and Assistant 3 cannot be assigned to task B. Find the optimal assignment schedule.

3. A company has four zones open and four salesmen available for assignment. The zones are not equally rich in their sales potentials. It is estimated that a typical salesman operating in each zone would bring in the following annual sales in ₹:

Zone: A: 1,26,000: Zone B:1,05,000; Zone C: 84,000; Zone D: 63,000.

The four salesmen are also considered to differ in ability. It is estimated that working under the same condition their sales per year would be proportionately as follows:

Salesman P:7 ; Salesman Q:5; Salesman R:5; Salesman S:4. If the criterion is maximum expected total sales, the intuitive answer is to assign the best salesman to the richest zone, the next best to the second richest zone and so on. Verify this by the method of assignment.

4. Average time taken by an operator on a specific machine is tabulated below. The management is considering to replace one of the old machines by a new one and the estimated time (Hour) for operation by each operator on the new machine is also indicated.

Machines							
Operator	M_1	M_2	M_3	M_4	M_5	M_6	New
01	2	3	2	1	4	5	6
02	4	4	6	3	2	5	1
03	6	10	8	4	7	6	1
04	8	7	6	5	3	9	4
05	7	3	4	5	4	3	12
06	5	5	6	7	8	1	6

Find out an allocation of operators to the old machines to achieve a minimum operation time.

(b) Reset the problem with the new machine and find out the allocation of the operators to each machine and comment on whether it is advantageous to replace an old machine to achieve a reduction in operating time only.

(c) How will the operators be reallocated to the machines after replacement?

5. An air-line operates seven days a week has time-table shown below. Crews must have a minimum layover (rest) time of 5 hrs, between flights. Obtain the pair of flights that minimizes layover time away from home. For any given pair the crews will be based at the city that result in the smaller layover.

Delhi-Jaipur			Jaipur-Delhi		
Flight No.	Depart	Arrive	Flight No.	Depart	Arrive
1	7.00 AM	8.00 AM	101	8.00 AM	9.15 AM
2	8.00 AM	9.00 AM	102	8.30 AM	9.45 AM
3	1.30 PM	2.30 PM	103	12.00 NOON	1.15 PM
4	6.30 AM	7.30 PM	104	5.30 PM	6.45 PM

for each pair, mention the town where the crews should be based.

[Hints – First of all construct a layover matrix showing times between flights when the crew is based in Delhi. For the sake of simplicity assume 15 minutes = 1 unit.

Flight	101	102	103	104
1	96	98	112	38
2	92	94	108	34
3	70	72	86	108
4	50	52	66	88

Since, the crew must have a minimum layover of 5 hrs between flighhts

The layover time between flights 1 and 101 will be 24 hrs (96 units) from 8.00 AM to 8.00 AM next day i.e.

flight 1 arrives Jaipur at 8.00 am and leaves Jaipur 8.00 am next day because the minimum layover is 5 hrs between flights and other flights are there in between. So flight will be there next day only.

Flight 1 to 102 will be (98 units) 8.00 am arrives at Jaipur and leaves Jaipur at 8.30 am next day = 24 hrs + 30 minutes

Flight 1 to 103 will be (112 units) 8.00 am arrives at Jaipur and leaves Jaipur at 12.00 noon next day = 24 hrs + 4 hrs = 112 units

Flight 1 to 104 will be (38 units) 8.00 am arrives at Jaipur and leaves Jaipur at 5.30 pm on the same day = 9 hrs + 30 min = 38 mins

Similarly timings for other flights are also computed.

Next the layover matrix is formed for the situation when the crew is based at Jaipur and shown below.

Flight	101	102	103	104
1	87	85	71	49
2	91	89	75	53
3	113	111	97	75
4	37	35	21	95

Now a table for minimum layover times between the flights is constructed using the data of the above two tables. It is shown below. [Data marked with (*) denote the crew is based at Jaipur.

Flight	101	102	103	104
1	87*	85*	71*	38
2	91*	89*	75	34
3	70	72	86	75
4	37*	35*	21*	88

Using Hungarian Method the problem is solved now and the optimal assignment is given below.

Flight 1 – 103, Flight 2 – 104, Flight 3 – 101 and Flight 4 – 102]

Answer:

1. Assignments are – 1 to A, 2 to Dummy, 3 to B and 4 to C   Optimum assignment = 30 units

   Alternative solution – 1 to C, 2 to Dummy, 3 to B and 4 to A

2. Assignments are – 1 to B, 2 to D, 3 to A and 4 to C   Minimum total time required = 18 hours

   Alternative solution – 1 to C, 2 to D, 3 to A and 4 to B

3. Assignments are – Salesman P to Zone A, Salesman Q to Zone B, Salesman R to Zone C & Salesman S to Zone D   Maximum Sales = ₹ 297000

4. (a)

Operation	Machine	Time (hours)
01	M_3	2
02	M_1	4
03	M_4	4
04	M_5	3
05	M_2	3
06	M_6	1
Minimum Total Operation Time		**17**

(b)

Operation	Machine	Time (hours)
01	M_1	2
02	M_4	3
03	New	1
04	M_5	3
05	M_2	3
06	M_6	1
Minimum Total Operation Time		**13**

In place of M_3 New machine should be installed.

References:

- Kapoor V. K. – Operations Research (For Managerial Decision Making) Sultanchand & Sons
- Kapoor V.K. – Problems and Solutions in Operations Research, Sultanchand & Sons
- Kanti Swarup, Gupta and Manmohan – Operations Research

Game Theory 9

SLOB Mapped against the Module

To equip oneself with application-oriented knowledge of Game Theory to facilitate management decisions for optimisation through resource allocation, managing competition, work scheduling and managing cost overrun, demand estimation, production and cost analysis etc.

Module Learning Objectives

After studying this module, the students will be able to:

- Understand situations where decisions are to be made under conditions of conflict.
- Understand the different procedures used in the selection and execution of various strategies which result in winning the Game.

Game Theory

9

In decision making often we come across situations where two or more opposing parties are seen to have conflicting interests. Action of one depends on the action taken by the opponent. Any Military Operation is an example of such a conflicting situation. Each participant of the operation takes all possible measures to prevent the opponent from succeeding. Situations in the field of Economics, particularly when there is free competition belong to the class of conflicts, too. Firms trying to maintain their market share, work in a conflicting or competitive environment and any move by one is suitably counter moved by the other. To increase the market share if one firm takes a strategy of reducing the selling price of its product by giving some discount then the other might take a strategy to redesign its product and increase its value at a price lower than the competitor.

To analyse such conflicting situations, some special mathematical model called "Game Theory" is used. It was first developed, to solve problems in economics, by Hungarian born American mathematician John von Neumann and his Princeton University colleague Oskar Morgenstern, a German born American economist in the year 1944. They observed that economics is much like a game, wherein players anticipate each other's moves. They named it Game Theory which is somewhat of a misnomer because it does not share the fun or frivolity associated with games.

Game Theory may be defined as a type of Decision Making situation when two or more intelligent and rational opponents are involved under conditions of conflict and competition. It is a type of Decision Theory in which one's choice of action is determined after taking into account all possible alternatives available to the opponent participating in the same game. Game Theory does not insist on how a game should be played but tells the procedure and principles by which action should be selected. 'Game' is defined as an activity between two or more participants according to a set of rules, at the end of which each participant either gets some benefit or suffers some loss.

Basic Terms

A. **Player –** A participant is called a Player.

B. **Play –** A Play of the game is said to occur when each Player has chosen a course of action.

C. **2 Person Game –** If the number of Players in a Game is two then it is called 2 Person Game. The term Person refers to an individual or a group aiming at a particular objective.

D. **N Person Game –** If the number of Players in a Game is N (where $N > 2$) then it is called N Person Game.

E. **Zero Sum Game –** If the sum of the amounts won by all winners is equal to that lost by all losers then the game is called Zero Sum Game. In other words, sum of the gains and losses in such a game is zero. When there are only two players participating in a game which has resulted in zero sum then it is called 2 Person Zero Sum Game. For a game with zero sum and N participants, we call it as N Person Zero Sum Game.

An example of 2 Person Zero Sum Game is the competition of two firms who are trying to increase their share of the market. Here gain of the market share of one will almost be equal to the other's loss of the market share.

Two Person Zero Sum Games are also called rectangular game because their payoff matrix is in the rectangular form.

F. **Non Zero Sum Game –** If the sum of the gains or losses in a game is not equal to zero then it is called a Non Zero Sum Game. An example of such a game is the competition between two firms for increasing their respective market shares through intensive advertising campaigns. Due to the advertisement of their product, both the firms might gain market share which may not be of equal magnitude, but at the same time gain of one is not exactly equal to the loss of the other. Hence, sum of the amounts is not zero.

G. **Strategy:** It is the predetermined rule by which a Player while playing decides the course of action from his own list of courses of action. There are two types of strategies – Pure and Mixed.

H. **Pure Strategy –** If a Player knows exactly what the other Player is going to do, a deterministic situation is obtained. The objective is to maximize the gain. Thus, it is a decision in advance of all plays always to choose a particular course of action. A pure strategy is usually represented by a number with which the course of action is associated.

I. **Mixed Strategy –** If a Player is guessing as to which activity is to be selected by the other on any particular occasion, a probabilistic situation is obtained. The objective in this case is to maximize the expected gain. Thus, it is a decision, in advance of all plays to choose a course of action for each play in accordance with some particular probability distribution. When a player decides in advance to use his available courses of action in some fixed proportion, he is said to use mixed strategy. In other words we can say that the Mixed Strategy is a selection among Pure Strategies with some fixed probabilities.

J. **Payoff –** The outcome of playing the game is known as Payoff. It is the quantitative measure of satisfaction a Player gets at the end of each play. For a business situation, the measure of satisfaction mentioned above could be increase in profits, Expansion in actual market share etc. In other words, it is the net gain a course of action or strategy brings to a player for any counter course of action or strategy of the competitor.

K. **Payoff Matrix –** This is a tabular representation showing the outcomes or payoffs corresponding to different strategies of the participating Players. Since a Game involves at least two Players, the table referred above always forms a matrix with some rows (m, say) and columns (n, say). Rules of a Payoff Matrix are –

- Rows denote the activities or courses of action available to Player A who is considered as the maximising player.
- Columns denote the activities or courses of action available to Player B who is the minimising player.
- Figure shown in the cell x_{ij} denotes payment to A when he chooses the activity i against B's choice of activity j.
- For a Two Person Zero Sum Game, any cell entry of the Player B's payoff matrix will be negative of the corresponding cell entry in the Player A's payoff matrix, so that the sum of the payoffs of the two Players is ultimately zero.

The following table is an example of a Payoff Matrix of a Two Person Zero Sum Game which says that two firms are competing for business with the mentioned strategies so that one's gain is another's loss:

Strategies of Firm A	Strategies of Firm B		
	No advertising	Medium advertising	Heavy advertising
No advertising	10	5	-2
Heavy advertising	16	14	10

Here a positive payoff denotes gain to the maximising player i.e. Firm A (shown as Row) and loss to the minimising player i.e. Firm B (shown as Column). If Firm A chooses strategy "No advertising" and Firm B chooses strategy "Medium advertising" then gain of A will be 5 and loss of B will also be 5.

L. **Optimal Strategy:** A course of action which puts the Player in the most preferred position, irrespective of the strategy of his competitors, is called Optimal Strategy. Not opting for this strategy will result in decreased payoff of the player.

M. **Value of the Game:** It is the expected payoff of play when all the players of the game follow their optimal strategies. The game is called Fair if the value of the game is zero and Unfair if it is non-zero.

Assumptions in a Game

1. The players act rationally and intelligently.
2. Each player has a finite set of strategies available to him.
3. The players attempt to maximise gains and minimise losses.
4. All relevant information is available to each player.
5. The players take individual decisions without direct communication with each other.
6. The players select their strategies simultaneously.
7. The payoff is fixed and determined in advance.

Solution of Pure Strategy Games with Saddle Point

Pure Strategy Games are solved using Maximin – Minimax criteria. The maximising player (whose strategies are shown along the rows of the Payoff Matrix) arrives at his optimal strategy on the basis of Maximin criteria and the minimising player (whose strategies are shown along the columns of the Payoff Matrix) follows Minimax criteria. The game is solved when Maximin and Minimax values are equal.

Maximin value is determined as follows –

(i) Find minimum value in each row of the given payoff matrix. This denotes minimum possible gain against each strategy of the Maximising Player.

(ii) Maximin value is the maximum of these minimum values.

Minimax value is determined as follows –

(i) Find maximum value in each column of the given payoff matrix. This denotes maximum possible loss against each strategy of the Minimising Player.

(ii) Minimax value is the minimum of these maximum values.

Saddle Point is said to exist when the Maximin and Minimax values are equal. Thus, Saddle Point is the position of such an element in the payoff matrix, which is minimum in its row and maximum in its column. The Saddle Point is the solution or Value of the game. The strategies of the two players corresponding to the Saddle Point are their optimal strategies. If there is more than one Saddle Point then more than one solution will be possible corresponding to each Saddle Point.

Following Payoff Matrix is used to illustrate 2 Person Zero Sum Pure Strategy Game:

		Strategies of Player B		Row Minimum
		B_1	B_2	
Strategies of Player A	A_1	9	2	2
	A_2	8	6	**6 = Maximin**
	A_3	6	4	4
Column Maximum		9	**6 = Minimax**	

In the table above, A is the maximising player with strategies represented along the rows and B is the minimising player with strategies represented along the columns.

Suppose the player A starts the game knowing fully well that whatever strategy he adopts B will select that particular counter strategy which will minimise the payoff to A. If A selects A_1 then B will definitely select B_2 so that A gets minimum possible gain i.e. 2 under the situation. Similarly if A chooses A_2 then B will go for B_2 and so on. Thus, A wants to maximise his gain which is possible by going for the maximum value among the Row minimums or the Maximin value. Similarly, B wants to minimise his loss which is the minimum among the Column maximums or the Minimax value.

We observe here that both Maximin and Minimax values are equal to 6. Hence there exists a Saddle Point. Also this value corresponds to the cell A_2B_2. That means the Optimal strategy for the Player A is A_2 and that for the Player B is B_2. Value of the Game is 6 for A and -6 for B which means the game is Zero Sum.

Principle of Dominance

According to the Principle of Dominance if any strategy of a player dominates over his another strategy in all conditions then the later can be ignored because it will not affect the solution of the game. A strategy dominates over the other only if it is preferable over the other in all conditions. From the gainer's point of view, if a strategy gives more gain than another strategy for all strategies of the loser, then the first strategy dominates over the other and the second one can be ignored altogether. Similarly from the loser's point of view, if a strategy involves lesser loss than the other in all conditions, the second one can be omitted without affecting decision. So determination of superior or inferior strategy depends upon the objective of the player. Since each player has to select his best strategy, the inferior strategies can be eliminated. In other words, ineffective rows and columns can be deleted from the given payoff matrix so that its size is reduced.

For deleting the ineffective rows and columns, the following Rules are used –

Rule 1 – If all the elements of a row (say ith row) of a payoff matrix are less than or equal to the corresponding elements of another row (say jth row) then the Maximising Player will never choose the ith strategy. In other words ith strategy is dominated by the jth strategy.

Rule 2 – If all the elements of a column (say pth column) of a payoff matrix are more than or equal to the corresponding elements of another column (say qth column) then the Minimising Player will never choose the pth strategy. In other words, pth strategy is dominated by the qth strategy.

Rule 3 – A pure strategy may be dominated if it is inferior to average of two or more other pure strategies. If all the elements of a row are less than or equal to the average of the corresponding elements of two or more other rows then this row is said to be dominated by the other group of rows for which average is computed. Similar concept is also applicable for column with the exception of having its elements more than the average of the corresponding elements of two or more columns.

Principle of Dominance can be applied to both Pure Strategy as well as Mixed Strategy problems. Its basic objective is to reduce the size of the given Payoff Matrix. Aim should always be made to get a (2 × 2) matrix by using this Principle.

Solution of Mixed Strategy Games

Any problem of Game without a Saddle Point is considered to be the problem of Mixed Strategy. In such cases both players will use various strategies with certain probabilities to optimize. Unlike Pure Strategy problems (where a single strategy will certainly be the optimum one) here we need to find out the probabilities of various strategies of both the players as well as expected value of the game. Games with Mixed Strategy are solved by the following methods depending on the size of the Payoff Matrix.

- (2 × 2) Game – Odds Method or Arithmetic Method
- Dominance Method (applicable for m × n Payoff Matrix convertible to 2 × 2 Payoff Matrix by application of Rules of Dominance)
- (2 × n) and (m × 2) Game – Graphical Method

1. Odds Method

Odds Method is applicable if and only if the Payoff Matrix is of size (2 × 2). Odds are nothing but the magnitude (i.e. without sign or ignoring negative sign, if any) of the differences of the elements of various rows as well as columns. Method of calculating Odds is given below –

1. Find out magnitude of difference in the values of cell (1,1) and (1,2) of the 1st Row and place it against the 2nd Row.
2. Compute magnitude of the difference in the cell entries of (2,1) and (2,2) of the 2nd Row and put it against the 1st Row.
3. Compute magnitude of the difference in the cell entries of (1,1) and (2,1) of the 1st Column and put it below the 2nd Column.
4. Compute magnitude of the difference in the cell entries of (1,2) and (2,2) of the 2nd Column and put it below the 1st Column.
5. Ensure that the sum of the differences calculated for the Rows is equal to that for the columns. In other words Sum of the differences calculated in steps (1) and (2) should be equal to that calculated in steps (3) and (4).

Note – Only the magnitude of the differences should be taken into account ignoring the negative signs, if any.

		Strategies of Y		
		Y_1	Y_2	ODDs
Strategies of X	X_1	a_1	a_2	$(b_1 - b_2)$
	X_2	b_1	b_2	$(a_1 - a_2)$
	ODDs	$(a_2 - b_2)$	$(a_1 - b_1)$	

Probabilities of X as well as Y taking different strategies are calculated by using the following formulae –

$P(X_1) = (b_1 - b_2) \div [(b_1 - b_2) + (a_1 - a_2)]$ and $P(X_2) = (a_1 - a_2) \div [(b_1 - b_2) + (a_1 - a_2)]$

$P(Y_1) = (a_2 - b_2) \div [(a_2 - b_2) + (a_1 - b_1)]$ and $P(Y_2) = (a_1 - b_1) \div [(a_2 - b_2) + (a_1 - b_1)]$

Value of the Game is determined using the formula:- $v = [a_1(b_1 - b_2) + b_1(a_1 - a_2)] \div [(b_1 - b_2) + (a_1 - a_2)]$

[Note: $P(X_1) + P(X_2) = 1$ and $P(Y_1) + P(Y_2) = 1$. So once $P(X_1)$ is calculated, $P(X_2)$ can always be calculated as complement of $P(X_1)$ instead of going for the formula. Similar is the case for $P(Y_2)$.]

2. Dominance Method

Dominance Method is applied for reducing the size of (m × n) Payoff Matrix (when either one of m and n or both m and n are greater than 2) when there exist no Saddle Point. The aim is to get (2 × 2) Matrix, so that Odds Method can be applied to find the Probabilities and the Value of the Game as described above. It can be mentioned that the strategies which are dominated by the others and ultimately ignored will not be used by the players and hence their probabilities will be zero.

Illustration 1

Solve the Game with the Payoff Matrix $\begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix}$

Solution:

Let the given Game is played by the Players A and B with A (the maximising player) having strategies A_1 and A_2 represented along the rows and B (the minimising player) having strategies B_1 and B_2 represented along the columns. So the given Payoff Matrix can be written as follows –

		Strategies of B		
		B_1	B_2	Row Min.
Strategies of A	A_1	1	5	1
	A_2	4	2	2 = Maximin
	Column Max.	4 = Minimax	5	

Maximin value (2) ≠ Minimax value (4). Thus, Saddle Point does not exist. So this is a problem of Mixed Strategy with (2 × 2) Payoff Matrix.

We apply Odds Method to solve the problem. Odds are calculated as follows –

		Strategies of B		
		B_1	B_2	ODDs
Strategies of A	A_1	$1 = a_1$	$5 = a_2$	$b_1 - b_2 = 4\text{-}2 = 2$
	A_2	$4 = b_1$	$2 = b_2$	$a_1 - a_2 = 1\text{-}5 = 4$
	ODDs	$a_2 - b_2 = 5\text{-} 2 = 3$	$a_1 - b_1 = 1\text{-}4 = 3$	

[**Note**: Though $(a_1 - b_1) = 1 - 4 = -3$, but here it has been taken +3 as per the concept of Odds, similar is the case for $(a_1 - a_2)$]

Probabilities of A and B taking their different strategies are calculated as follows –

$P(A_1) = (b_1 - b_2) \div [(b_1 - b_2) + (a_1 - a_2)] = 2 / [2 + 4] = 2/6 = 1/3$

$P(A_2) = (a_1 - a_2) \div [(b_1 - b_2) + (a_1 - a_2)] = 4 / [2 + 4] = 4/6 = 2/3$

$P(B_1) = (a_2 - b_2) \div [(a_2 - b_2) + (a_1 - b_1)] = 3 / [3 + 3] = 3/6 = 1/2$

$P(B_2) = (a_1 - b_1) \div [(a_2 - b_2) + (a_1 - b_1)] = 3 / [3 + 3] = 3/6 = 1/2$

Value of the Game = $v = [a_1(b_1 - b_2) + b_1(a_1 - a_2)] \div [(b_1 - b_2) + (a_1 - a_2)] = [1\times2 + 4\times4] \div [2 + 4] = 18/6 = 3$

So A chooses his strategies (A_1, A_2) with probabilities (1/3, 2/3) & B chooses his strategies (B_1, B_2) with probabilities (1/2, 1/2) and Value of the Game = 3

[**Note:** Calculated Value of the Game is the Expected Gain of A which is same as the Expected Loss of B]

Illustration 2

The Management of a company is negotiating with its Union for revision of hourly wages of its employees. The Management deployed a Consulting Firm who has prepared a payoff matrix for the purpose which indicates the additional hourly cost (in ₹) to the company. It is shown below: you being a part of the Consulting Firm have to assist the Management in selecting the best strategy. What is the value of the game? How is it going to affect the company's cost?

Management's Strategies	Strategies of the Union			
	U_1	U_2	U_3	U_4
M_1	2.50	2.70	3.50	– 0.20
M_2	2.00	1.60	0.80	0.80
M_3	1.40	1.20	1.50	1.30
M_4	3.00	1.40	1.90	0

Solution:

As the Management's objective is to minimise the cost, they can be considered as the Minimising Player and the Union as the Maximising Player in this problem of Game. Thus, to solve the problem we have to recast the given Payoff Matrix by transposing it as below:-

Strategies of the Union	Management's Strategies				Row Minimum
	M_1	M_2	M_3	M_4	
U_1	2.50	2.00	1.40	3.00	1.40 = Maximin
U_2	2.70	1.60	1.20	1.40	1.20
U_3	3.50	0.80	1.50	1.90	0.80
U_4	-0.20	0.80	1.30	0	-0.20
Column Maximum	3.50	2.00	1.50 = Minimax	3.00	

Maximin value (1.40) ≠ Minimax value (1.50). Thus, Saddle Point does not exist. So this is a problem of Mixed Strategy. Since the matrix is not a (2×2) Matrix, Dominance Rules are applied to reduce its size to make it a (2×2) Matrix.

As all the elements of the 3rd Row of the above matrix are either greater than or equal to the corresponding elements of the 4th Row, the 3rd Row can be considered to dominate the 4th. So the 4th Row is ignored and the new matrix is shown below.

Strategies of the Union	Management's Strategies			
	M_1	M_2	M_3	M_4
U_1	2.50	2.00	1.40	3.00
U_2	2.70	1.60	1.20	1.40
U_3	3.50	0.80	1.50	1.90

Again all the elements of the 1st Column are greater than the corresponding elements of the 2nd Column, the 1st Column is dominated by the 2nd Column. Hence the 1st Column is ignored and the new matrix is shown below.

Strategies of the Union	Management's Strategies		
	M_2	M_3	M_4
U_1	2.00	1.40	3.00
U_2	1.60	1.20	1.40
U_3	0.80	1.50	1.90

All the elements of the 3rd Column (i.e. for Strategy M_4) of this matrix are more than the corresponding elements of 2nd Column (i.e. for Strategy M_3). Hence M_4 is dominated by M_3 and ignored. The new matrix is shown below.

Strategies of the Union	Management's Strategies	
	M_2	M_3
U_1	2.00	1.40
U_2	1.60	1.20
U_3	0.80	1.50

Again all the elements of the 1st Row (for strategy U_1) are greater than the corresponding elements of the 2nd Row (for strategy U_2). So U_2 is dominated by U_1 and ignored. The new matrix is shown below.

Strategies of the Union	Management's Strategies	
	M_2	M_3
U_1	2.00	1.40
U_3	0.80	1.50

This is a (2 × 2) Matrix. Now the problem of Game is solved by using Odds Method. Odds are calculated as below.

Strategies of the Union	Management's Strategies		ODDs
	M_2	M_3	
U_1	$2.00 = a_1$	$1.40 = a_2$	$b_1 - b_2 = 0.80 - 1.50 = 0.70$
U_3	$0.80 = b_1$	$1.50 = b_2$	$a_1 - a_2 = 2.00 - 1.40 = 0.60$
ODDs	$a_2 - b_2 = 1.40 - 1.50 = 0.10$	$a_1 - b_1 = 2.00 - 0.80 = 1.20$	Sum of the ODDs = 1.30

Probabilities of the Union and the Management taking their different strategies are calculated as follows –

$P(U_1) = (b_1 - b_2) \div [(b_1 - b_2) + (a_1 - a_2)] = 0.70 / [0.70 + 0.60] = 0.70/1.30 = 7/13$

$P(U_3) = (a_1 - a_2) \div [(b_1 - b_2) + (a_1 - a_2)] = 0.60 / [0.70 + 0.60] = 0.60/1.30 = 6/13$

$P(M_2) = (a_2 - b_2) \div [(a_2 - b_2) + (a_1 - b_1)] = 0.10 / [0.10 + 1.20] = 0.10/1.30 = 1/13$

$P(M_3) = (a_1 - b_1) \div [(a_2 - b_2) + (a_1 - b_1)] = 1.20 / [0.10 + 1.20] = 1.20/1.30 = 12/13$

Value of the Game $= v = [a_1(b_1 - b_2) + b_1(a_1 - a_2)] \div [(b_1 - b_2) + (a_1 - a_2)] = [2.00\times0.70 + 0.80\times0.60] \div [0.70+ 0.60]$

$= [1.40 + 0.48] /1.30 = 1.88/1.30 = 1.45$

So the Union chooses its Strategies U_1, U_2, U_3 & U_4 with probabilities (7/13, 0, 6/13, 0) and the Management chooses its Strategies M_1, M_2, M_3 & M_4 with probabilities (0, 1/13, 12/13, 0).

Expected Gain to the Union is ₹ 1.45 and the corresponding Loss to the Management is ₹ 1.45.

Thus, the hourly cost of the company will increase by ₹ 1.45

Illustration 3

Solve the Game using Dominance Principle $\begin{bmatrix} 1 & 3 & 12 \\ 8 & 6 & 2 \end{bmatrix}$

Solution:

Let the given Game is played by the Players A and B with A (the maximising player) having strategies A_1, A_2 and A_3 represented along the rows and B (the minimising player) having strategies B_1, B_2 and B_3 represented along the columns. So the given Payoff Matrix can be written as follows –

	Strategies of B		
Strategies of A	B_1	B_2	B_3
A_1	15	2	3
A_2	6	5	7
A_3	-7	4	0

All the elements of Row A_3 are less than the corresponding elements of Row A_2. So A_3 is dominated by A_2. Hence it is ignored and deleted. The new matrix is given below.

Strategies of A	Strategies of B		
	B_1	B_2	B_3
A_1	15	2	3
A_2	6	5	7

Here all the elements of B_3 are more than the corresponding elements of B_2. Hence B_3 is dominated by B_2 and ignored to get the new matrix below.

	Strategies of B		
Strategies of A	B_1	B_2	Row Min.
A_1	15	2	2
A_2	6	5	**5**
Column Max.	15	**5**	

Maximum among the Row minimums = 5 = Maximin value and Minimum among the Column maximums = 5 = Minimax value. As, Maximin and Minimax values are equal, there exists a Saddle Point. It occurs at the cell A_2B_2.

Hence optimal strategies of A and B are respectively A_2 and B_2. Also value of the Game = 5

[NOTE – This is a problem of Pure Strategy and could have been solved without the use of Dominance Rules, but the question has specifically asked for the usage of Dominance Rules. So the same is used.]

Illustration 4

Joy Givers and Milan Toys are the two toy manufacturers who always compete with each other to increase their respective market shares. For both the companies the Marketing team work with close coordination with the Design team and always come out with attractive toys which are normally in great demand. To meet the demand, they have various strategic options like working for 8 hours a day, 12 hours a day, 16 hours a day, 24 hours a day, Subcontracting etc. which will ultimately increase the market share. Joy Givers have decided not to go for all the above mentioned options and set up the following payoff matrix in which the percentage increase in market share is given against different strategies of Milan Toys

STRATEGIES of	Milan Toys			
Joy Givers	Working 8 hrs/day	Working 12 hrs/day	Working 16 hrs/day	Subcontracting
Working 12 hrs/day	8	10	9	14
Working 16 hrs/day	10	11	8	12
Working 24 hrs/day	13	12	14	13

Use Principle of Dominance to find the Optimal Strategies of the two manufacturers and the value of the Game.

Solution:

Joy Givers is the Maximising player with strategies represented along the rows and Milan Toys is the Minimising Player with strategies represented along the columns. For ease of representation we consider the respective strategies of Joy Givers as J_1, J_2 & J_3 and those of Milan Toys as M_1, M_2 & M_3.

STRATEGIES of	Milan Toys			
Joy Givers	M_1	M_2	M_3	M_4
J_1	8	10	9	14
J_2	10	11	8	12
J_3	13	12	14	13

All the elements of 4th Column are either greater than or equal to the corresponding elements of the 1st Column. So 4th Column's strategy (M_4) is dominated by the 1st Column's strategy (M_1). Hence M_4 is ignored. The new matrix is given below.

STRATEGIES	Milan Toys		
Joy Givers	M_1	M_2	M_3
J_1	8	10	9
J_2	10	11	8
J_3	13	12	14

All the elements of 1st Row are less than the corresponding elements of the 3rd Row. Thus, strategy of 1st Row i.e. J_1 is dominated by the strategy of the 3rd Row i.e. J_3 and ignored. The reduced matrix becomes –

STRATEGIES	Milan Toys		
Joy Givers	M_1	M_2	M_3
J_2	10	11	8
J_3	13	12	14

Apparently first two rules of dominance cannot be applied to either of the rows or columns of the above matrix, but if the average of the elements of the strategies M_2 and M_3 be taken then we get a matrix shown below.

STRATEGIES	Milan Toys	
Joy Givers	M_1	$[M_2 + M_3]/2$
J_2	10	$(11 + 8)/2 = 9.5$
J_3	13	$(12 + 14)/2 = 13$

So the elements of the strategy M_1 are either more or equal to the average of the corresponding elements of M_2 and M_3. Hence M_1 is dominated by M_2 and M_3. Thus, M_1 is deleted and the reduced matrix is as below.

STRATEGIES of	Milan Toys		Row Minimum
Joy Givers	M_2	M_3	
J_2	11	8	8
J_3	**12***	14	12 = Maximin
Column Maximum	12 = Minimax	14	

So Maximin value = 12 = Minimax value. Hence there exists a Saddle Point at the junction J_3M_2

Thus, optimal strategy of Joy Giver is J_3 that is "Working 24 hours /day" and that for Milan Toys is M_2 that is "Working 12 hours/day". Value of the Game = 12 (which means a 12% increase in market share for Joy Givers)

3. Graphical Method

Graphical Method is applied to solve (2 × n) and (m × 2) Game problems, when both m and n are more than 2. Since the optimal strategies for both the players assign non zero probabilities to the same number of pure strategies, it is obvious that if one player has only two strategies the other will also use two strategies. Graphical method facilitates to find out which of the two strategies can be used. When Rules of Dominance cannot be applied to a payoff matrix of size (2 × n) or (m × 2) then Graphical Method is used.

Following are the steps for solving a (2 × n) Game –

1. Draw two vertical lines 1 unit apart along a horizontal line to represent the axes $x_1 = 0$ and $x_1 = 1$ & mark a suitable scale on each one.
2. Take the values in the first Row of the Payoff Matrix and plot each one as a point on the scale of the vertical line $x_1 = 1$.
3. Take the values in the second Row of the Payoff Matrix & plot each one as a point on the scale of the vertical line $x_1 = 0$.
4. The point a_{1j} on the line $x_1 = 1$ should be joined to the point a_{2j} on the line $x_1 = 0$ to get a straight line.
5. Draw n such straight lines for j = 1, 2, 3,.......... n. Each of these lines represents the expected payoff of the maximising player (whose 2 strategies are represented by the rows) against n different strategies of the minimising player (whose strategies are represented by the columns).
6. Mark the lower envelope of the area obtained by drawing these n straight lines.
7. The highest point of the lower envelope is the Maximin point.
8. The straight lines passing through this Maximin point corresponds to the optimum strategies of the minimising player. All the other strategies of the minimising player should be ignored.
9. So now the desired (2 × 2) payoff matrix is obtained, the Game can be solved using the method of Odds.

The steps for solving (m × 2) Game are almost similar to those of (2×n) Game. The main difference with respect to the previous case lies in the fact that we have to consider the values in the First and the second Columns and plot them as points on the two axes $x_1 = 0$ and $x_1 = 1$ so that m straight lines can be drawn representing expected payoff of the minimising player against the strategies of the maximising player. Now the upper envelope of the common area bounded by the straight lines should be marked. Lowest point of this envelope represents Minimax point. The strategy lines which intersect to give the Minimax point are the optimum strategies of the maximising player. All the other strategies of the maximising player should be ignored to give a (2×2) payoff matrix which can be solved by using the method of Odds.

[**Note**: Problems of Game, with Payoff Matrix of size (m×n) when both m & n > 2, are solved using Simplex Method of Linear Programming]

Illustration 5

Solve the Game represented by the payoff matrix :- $\begin{bmatrix} 1 & 3 & 12 \\ 8 & 6 & 2 \end{bmatrix}$

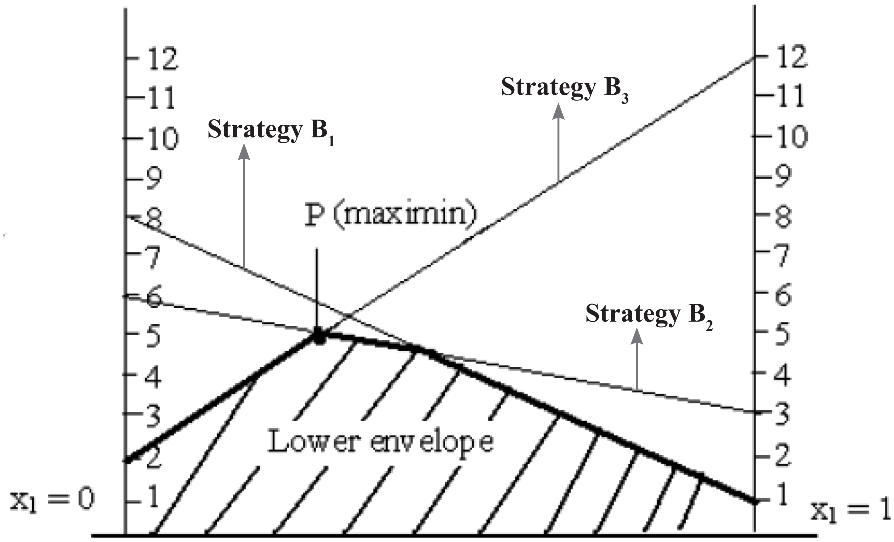
Solution:

Let the given Game is played by the Players A and B with A (the maximising player) having strategies A_1 and A_2 represented along the rows and B (the minimising player) having strategies B_1, B_2 and B_3 represented along the columns. So the given Payoff Matrix can be written as follows –

	Strategies of B			Row Minimum
Strategies of A	B_1	B_2	B_3	
A_1	1	3	12	1
A_2	8	6	2	2 = Maximin
Column Maximum	8	6 = Minimax	12	

Maximin value (2) ≠ Minimax value (6). Thus, Saddle Point does not exist. So this is a problem of Mixed Strategy. Since the matrix is not a (2×2) Matrix, we check the possibility of applying Dominance Rules to reduce its size to (2×2) Matrix, but it is observed that the Dominance Rules are also not suitable for reducing the size of the given matrix. Hence we go for solving the problem using Graphical Method suitable for (2×n) matrix because the given matrix is (2×3)

As shown below two vertical lines are drawn on a horizontal line 1 unit apart to represent the axes $x_1 = 0$ & $x_1 = 1$ and marked them to a scale. Now the values 1, 3 and 12 of the 1st Row of the given matrix are plotted as points on the axis $x_1 = 1$ and the values 8, 6 and 2 of the 2nd Row are plotted as points on the axis $x_1 = 0$. Then the pair of points 12 & 2, 3 & 6 and 1 & 8 are joined with the help of straight lines. These lines represent the expected payoff of Player A against the strategies B_3, B_2 & B_1 respectively of Player B. The lower envelope of the area bounded by these lines and the axes is shaded as shown. Highest point P of this envelope is the Maximin point. As P is the point of intersection of A's expected payoff lines against strategies B_2 & B_3 of B, we can say that B will opt for these two strategies and ignore strategy B_1.



So the reduced payoff matrix is given as follows –

Strategies of A	Strategies of B		
	B_2	B_3	ODDs
A_1	$3 = a_1$	$12 = a_2$	$b_1 - b_2 = 6 - 2 = 4$
A_2	$6 = b_1$	$2 = b_2$	$a_1 - a_2 = 3 - 12 = 9$
ODDs	$a_2 - b_2 = 12 - 2 = 10$	$a_1 - b_1 = 3 - 6 = 3$	Sum of the ODDs = 13

Probabilities of A and B taking their different strategies are calculated as follows –

$P(A_1) = (b_1 - b_2) \div [(b_1 - b_2) + (a_1 - a_2)] = 4 / [4 + 9] = 4/13$, $P(A_2) = 1 - P(A_1) = 1 - 4/13 = 9/13$

$P(B_2) = (a_2 - b_2) \div [(a_2 - b_2) + (a_1 - b_1)] = 10 / [10 + 3] = 10/13$, $P(B_3) = 1 - P(B_2) = 1 - 10/13 = 3/13$

Value of the Game = $v = [a_1(b_1 - b_2) + b_1(a_1 - a_2)] \div [(b_1 - b_2) + (a_1 - a_2)] = [3\times4 + 6\times9] \div [4 + 9] = 66/13$

So A chooses the strategies with probabilities (4/13, 9/13) and B chooses the strategies with probabilities (0, 10/13, 3/13).

Limitations of Game Theory

1. The assumption that the players have the knowledge about their own and the opponent's payoffs is unrealistic. He can only make a guess of his own and the opponent's strategies.
2. As the number of players increase in the game, the analysis of the strategies becomes increasingly complex and difficult. In practice there are many firms in an oligopoly situation where game theory is not useful.
3. The assumptions of Pure Strategy game show that the players are risk averse and have complete knowledge about each other's strategies. This is impractical.
4. Rather than each player in an oligopoly situation working under uncertain conditions, the players will allow each other to share the secrets of business in order to work out a collusion. So the mixed strategies are also not very useful.

EXERCISE

A. Theoretical Questions:

Multiple Choice Questions

1. Two person zero sum game means that
   (a) The sum of losses of one player is equal to the sum of the gains of the other
   (b) The sum of losses of one player may not be equal to the sum of the gains of the other
   (c) No player gains or loses
   (d) None of the above

2. Game theory models are classified by the
   (a) Number of players
   (b) Sum of all payoffs
   (c) Number of strategies
   (d) All of these

3. A game is said to be unfair if
   (a) Upper and lower values of the game are not equal
   (b) Upper and lower values of the game are equal and the sum is zero
   (c) Option (a) is correct but not Option (b)
   (d) Option (b) is correct but not Option (a)

4. What happens when the maximin and minimax values of the game are equal?
   (a) No solution exists
   (b) Solution is mixed
   (c) Saddle point exists
   (d) None of these

5. A mixed strategy game can be solved by
   (a) Arithmetic method
   (b) Graphical method
   (c) Dominance method
   (d) All of these

6. The size of the payoff matrix of a game can be reduced by using the principle of
   (a) Game inversion
   (b) Rotation reduction
   (c) Dominance
   (d) Game transpose

7. The payoff value for which each player in a game always selects the same strategy is called the
   (a) Saddle point
   (b) Equilibrium point
   (c) Both option (a) and option (b)
   (d) None of the above

8. Games which involve more than two players are called
   (a) Conflicting games
   (b) Negotiable games
   (c) N person game
   (d) All of these

9. When the sum of the gains of one player is equal to the sum of the losses to another player then it is called
   (a) Fair game
   (b) Zero sum game
   (c) Both option (a) and option (b)
   (d) Only option (b) and not option (a)

10. When no saddle point is found in the payoff matrix of a game, the value of the game is found by
   (a) Reducing the size of the game to apply the odds method
   (b) Solving any one of the (2×2) sub game
   (c) Finding the average of the values of the payoff matrix
   (d) None of these

11. A saddle point exists when
   (a) Maximin value = Maximax value
   (b) Minimax value = Minimum value
   (c) Minimax value = Maximin value
   (d) Minimax value = Minimin value

12. In a pure strategy game
   (a) Any strategy can be selected arbitrarily
   (b) A particular strategy is selected by each player
   (c) Both players select their optimal strategy
   (d) None of these

13. In a mixed strategy game
   (a) No saddle point exists

(b) Each player selects the same strategy without considering the choice of the other

(c) Each player always selects the same strategy

(d) None of these

14. Game theory is the study of

(a) Selecting optimal strategies

(b) Resolving conflict between players

(c) Giving equal outcome to the participants

(d) None of the above

15. If the value of the game is zero, then the game is known as

(a) Fair strategy

(b) Pure strategy

(c) Pure game

(d) Mixed strategy

16. The games with saddle points are

(a) Probabilistic in nature

(b) Normative in nature

(c) Stochastic in nature

(d) Deterministic in nature

17. When the game is played on a predetermined course of action, which does not change throughout the game then it is known as

(a) Pure strategy game

(b) Fair strategy game

(c) Mixed strategy game

(d) Unsteady game

18. If the losses of Player A are the gains of Player B, then it is called

(a) Lump sum game

(b) Zero sum game

(c) Unfair game

(d) None of the above

19. Identify the incorrect one

(a) A game without saddle point is probabilistic

(b) Game with saddle point will have pure strategies

(c) Game with saddle point cannot be solved with dominance rule

(d) Game without saddle point has mixed strategies

20. In case there is no saddle point in a game then the game is
    (a) Deterministic game
    (b) Fair game
    (c) Mixed strategy game
    (d) Multi player game

21. When Minimax and Maximin criteria matches then
    (a) A fair game exists
    (b) An unfair game exists
    (c) Mixed strategy exists
    (d) Saddle point exists

22. When there is dominance in a game then
    (a) Least of the row ≥ Highest of another row
    (b) Least of the row ≤ Highest of another row
    (c) Every element in a row ≥ Corresponding element of another column
    (d) Every element in a row ≤ Corresponding element of another row

23. A game is played when
    (a) The manager gives signal
    (b) Each player chooses one of his courses of action simultaneously
    (c) The player who comes to the field first says he will start the game
    (d) When the latecomer starts the game

24. In a game the list of the courses of action with each player is
    (a) Finite
    (b) Infinite
    (c) Only 3
    (d) None of the above

25. When the game is having a saddle point then the method used to solve the game is
    (a) Linear Programming method
    (b) Minimax and Maximin criteria
    (c) Odds method
    (d) Graphical method

26. Linear Programming method should be used to determine the value of the game when the size of the payoff matrix is
    (a) 2×2
    (b) 3×4
    (c) m×2
    (d) 2×n

27. If there are more than two persons in a game then the game is known as
    (a) Non zero sum game
    (b) Open game
    (c) Multiplayer game
    (d) Big game

28. A competitive situation is known as
    (a) Competition
    (b) Marketing
    (c) Game
    (d) All the above

29. Which one of the following is an assumption of Game Theory?
    (a) All players act rationally and intelligently
    (b) The winner alone acts rationally
    (c) The loser acts intelligently
    (d) Both believes in luck

30. For the Payoff Matrix $\begin{pmatrix} -5 & -2 \\ 10 & 5 \end{pmatrix}$ the maximising player always uses
    (a) The first strategy
    (b) Average of the two strategies
    (c) The second strategy
    (d) All the above strategies

Answers:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a	d	a	c	d	c	a	c	d	a	c	c	a	a	a
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
d	a	b	c	c	d	d	b	a	b	b	c	c	a	c

State True or False

1. The name Game is derived from the fact that the end result always gives lot of fun to the players.
2. Strategies are the different courses of action of the players.
3. In Pure Strategy the objective of the maximizing player is to maximize the Gain, but in Mixed Strategy the objective is to maximize the Expected Gain.
4. Both Pure and Mixed Strategy problems can be solved by the Rules of Dominance.
5. A fair game results when the value of the game is zero.
6. Mixed Strategy Games are deterministic in nature.
7. Zero sum games always have two participants only.
8. A pure strategy may be dominated if it is inferior to average of two or more other pure strategies.
9. Columns of a payoff matrix represent the strategies of the maximising player
10. Graphical Method can be used for mixed strategy games having any size of payoff matrix.
11. In a game the players act rationally and intelligently.
12. In a (m×2) mixed strategy game the graphical method is used to find out the maximin point.
13. Equality of minimax and maximin values result in the existence of Saddle Point.
14. Optimal strategy is that course of action which puts a player in most preferred position.
15. Military operations are examples of game.

Answers:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F	T	T	T	T	F	F	T	F	F	T	F	T	T	T

Fill in the blanks

1. For the Payoff Matrix $\begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix}$ the ___ strategy of the minimising player is dominated by the third.
2. "Teesta" water distribution conflict between India and Bangladesh can be considered as a situation of _____.
3. Arithmetic Method is used to solve ______ strategy problems of Game.
4. Value of the Game $\begin{bmatrix} 2 & 3 \\ -5 & 5 \end{bmatrix}$ is ___.
5. As per the Rules of Dominance, a strategy of the maximising player is said to dominate another if all the elements of a row of the payoff matrix are ____ than or equal to the corresponding elements of the other.
6. Pure strategy games are ____ in nature.
7. Dominance principle has ____ rules.
8. Strategies of maximising player of a game are represented along the ____ of the payoff matrix.
9. For (2×n) matrix Mixed strategy Games ____ method is used to find the solution.
10. Columns of the payoff matrix of a game represent the strategies of the ____ player.

11. Game with payoff matrix (m×n) is solved using Simplex Method, if both m and n are greater than ___.
12. In a game, all relevant information is available to ____ player.
13. In (2×2) mixed strategy game, sum of the probabilities of the minimising player taking its two strategies is ___.
14. Pure strategy games always have a ____ point.
15. In problems of game negative payoff of the maximising player indicates a ___.
16. Odds Method is applicable to ____ matrix games only.
17. In practical business situations most of games have ___ than two players.
18. Multiplayer games are also known as ___ person game .
19. In graphical method for (2 × m) game the Maximin value is determined from the ____ point of the lower envelope.
20. A participant of any game is called a _____.

Answers:

1.	**Second**	**2.**	**Game**
3.	**Mixed**	**4.**	**2**
5.	**Greater**	**6.**	**Deterministic**
7.	**Three**	**8.**	**Row**
9.	**Graphical**	**10.**	**Minimising**
11.	**2**	**12.**	**Each**
13.	**1**	**14.**	**Saddle**
15.	**Loss**	**16.**	**(2×2)**
17.	**More**	**18.**	**N**
19.	**Highest**	**20.**	**Player**

B. Numerical Questions

Comprehensive Numerical Problems

1. Find the optimal strategies of the Players for the game having payoff matrix. What is the value of the Game?

Strategies	B_1	B_2	B_3	B_4
A_1	1	7	3	4
A_2	5	6	4	5
A_3	7	2	0	3

2. Solve the game $\begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix}$

3. Reduce the following game by Dominance Rules and Solve it.

		Strategies of Minimising Player				
		P	Q	R	S	T
Strategies of Maximising Player	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

4. Solve the game $\begin{pmatrix} 3 & -4 \\ 2 & 5 \\ -2 & 8 \end{pmatrix}$ using Graphical method

5. In the suburban area of a large city there are two stores Laxmi Bhandar and Goswami Stores who handle sundry goods. The total number of customers is equally divided between the two due to the fact that the price, quality of goods, services etc. of the two are at par. Assume that a gain of customers for Laxmi Bhander is a loss to Goswami Stores and vice versa. Both the stores plan to run annual sales during the festival period of the year. Sales are advertised through Social Media, Cable TV local channel and Printed Leaflets. Based on the past experience, Laxmi Bhander has prepared the following payoff matrix for the gain or loss in percentage of customers for its different strategies against various counter strategies of Goswami Stores.

	Strategies of Goswami Stores		
Strategies of Laxmi Bhander	**Printed leaflets**	**Cable TV**	**Social media**
Printed leaflets	30	40	– 80
Cable TV	0	15	– 20
Social media	90	20	50

Determine the optimal strategies and worth of such strategies for the stores. What is meant by the cell entry – 80 in the above payoff matrix?

6. Two competing firms (A and B) produce consumer goods of different kind. Among the products one is considered as their bread and butter in terms of the revenue generated. Both the firms are very cautious about the market share for this particular product and keep on doing advertisement campaigns throughout the year to retain the existing customers and also to attract the new ones. For this the marketing teams of both work round the clock and that of A developed data corresponding to varying degrees of advertisement. Same is given below:

   (a) If both the firms take same strategy to counter each other then their market share will be equal.

   (b) Against firm A's strategy of "No marketing" if B goes for "Medium marketing" then A's share of the market will be 40%. For the same strategy of A the market share will be 28% if B takes the strategy "Large marketing"

   (c) Against firm A's strategy of "Medium marketing" if B goes for "No marketing" then A's share of the market will be 70%. For the same strategy of A the market share will be 45% if B takes the strategy "Large marketing"

(d) Against firm A's strategy of "Large marketing" if B goes for "No marketing" then A's share of the market will be 75%. For the same strategy of A the market share will be 47.5% if B takes the strategy "Medium marketing"

Based on the above information prepare the Payoff Matrix. Solve the game problem to get the optimal strategies of the player A. What is the value of the game?

7. Using the data of the above problem prepare the Payoff Matrix for A when you are supplied with the following information.

(a) Selling price of the product = ₹ 4 per unit

(b) Variable cost of the product = ₹ 2.50 per unit

(c) Annual cost for Medium advertising = ₹ 5000

(d) Annual cost for Large advertising = ₹ 15000

(e) Annual sales volume of the product for Firm A = 30000 units

What advertising policy should firm A pursue?

Hints-

Find out the Annual sales volume, for different combination of strategies of A and B. As an example, Annual Sales volume corresponding to A's strategy of "Medium advertisement" and B's strategy of "Large advertisement" is 45% of 30000 = 13500 units

Calculate Annual Profit to the Firm A using the formula below for various combination of strategies of A and B.

Annual Profit = (Selling price – Variable cost) × Annual Sales volume – Annual cost of advertising

Example of this calculation is:-

For A's strategy of "Medium advertisement" and B's strategy of "Large advertisement" the Annual Profit of Firm A is (4 – 2.5) × 13500 – 5000 = ₹ 15250/-

When the Profit figures for all the combinations of strategies of A and B are calculated then the following payoff matrix is obtained.

	Strategies of B		
Strategies of A	**No advertising**	**Medium advertising**	**Large advertising**
No advertising	22500	18000	12600
Medium advertising	26500	17500	15250
Large advertising	18750	6375	7500

From the above matrix we find, against the various strategies of A, the minimum profit figures are as follows –

For No advertising – ₹ 12600

For Medium advertising – ₹ 15250

For Large advertising – ₹ 6375

Thus, to maximise the minimum profit, A should opt for Medium advertising and spend ₹ 5000 per annum

Answers:

1. Optimal strategies A_2 and B_3. Value of the game = 4
2. Strategies of the Maximising Player = (1/5, 4/5) & for the Minimising Player = (3/5, 2/5). Value of the game = 17/5
3. Optimal strategy for the Maximising Player is III and that for the Minimising Player is Q. Value of the game = 5
4. Optimal strategy of the Maximising Player (0.3, 0.7, 0) and for the Minimising Player (0.9, 0.1). Value = 2.3
5. Optimal strategies of Laxmi Bhander = (1/5, 0, 4/5) and for Goswami Stores = (0, 13/15, 2/15), Value of the game = 24

   Cell entry (– 80) means when Laxmi Bhander will take the strategy of distributing Printed Leaflets against the counter strategy of Goswami Stores of Social Media advertisement then they will lose 80% of their customer which will be gained by Goswami Stores.
6. The payoff matrix Showing A's market share is –

	Strategies of B		
Strategies of A	**No advertising**	**Medium advertising**	**Large advertising**
No advertising	50	40	28
Medium advertising	70	50	45
Large advertising	75	47.5	50

Probabilities of A's strategies are (0, 1/3, 2/3). Value of the game = 145/3 = 48.3

Thus, A can expect to have 48.3% market share.

Reference:

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